WMA11_P4(IAL)_Summer_2020_Q1

Solution

To prove that if n^3 is even, then n is even using the method of **proof by contradiction**, we follow these logical steps:

1. State the Assumption Assume the negation of the statement we wish to prove. We are given that n is an **integer** and n^3 is **even**. We assume, for the sake of contradiction, that n is **odd**.

2. Algebraic Representation of an Odd Integer If n is an odd integer, it can be expressed in the form:

$$n = 2k + 1$$

where k is an integer.

3. Calculate the Cube of n We now find the expression for n^3 by cubing the algebraic form of n:

$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= (2k)^3 + 3(2k)^2(1) + 3(2k)(1)^2 + 1^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \end{aligned}$$

4. Analyze the Parity of n^3 We can factor out a 2 from the first three terms of the resulting expression:

$$n^3 = 2(4k^3 + 6k^2 + 3k) + 1$$

Let $m = 4k^3 + 6k^2 + 3k$. Since k is an integer, m must also be an integer. Thus:

$$n^3 = 2m + 1$$

By definition, an integer of the form $2m + 1$ is an **odd** number.

5. Identify the Contradiction Our algebraic derivation shows that if n is odd, then n^3 must be odd. However, the initial premise of the problem states that n^3 is **even**.

- This is a contradiction because an integer cannot be both even and odd.

6. Conclusion Since the assumption that n is odd leads to a contradiction, the assumption must be false. Therefore, n must be even.

Final Statement: By **proof by contradiction**, if n^3 is even for an integer n, then n must be even.

WMA11_P4(IAL)_Summer_2020_Q2

Solution

1. Binomial Expansion of $(4-5x)^{-1/2}$

To expand $(4-5x)^{-1/2}$ using the **Binomial Theorem** for non-integer powers, we first factor out the constant term to transform the expression into the form $(1+u)^n$.

- **Step 1: Factor out the constant 4**

$$\begin{aligned}(4-5x)^{-1/2} &= \left[4\left(1-\frac{5}{4}x\right)\right]^{-1/2} \\ &= 4^{-1/2}\left(1-\frac{5}{4}x\right)^{-1/2} \\ &= \frac{1}{2}\left(1-\frac{5}{4}x\right)^{-1/2}\end{aligned}$$

- **Step 2: Apply the general binomial expansion formula** The expansion for $(1+u)^n$ is given by $1+nu+\frac{n(n-1)}{2!}u^2+\dots$ Here, $n=-1/2$ and $u=-5x/4$.

$$\begin{aligned}\left(1-\frac{5}{4}x\right)^{-1/2} &= 1+\left(-\frac{1}{2}\right)\left(-\frac{5}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{5}{4}x\right)^2+\dots \\ &= 1+\frac{5}{8}x+\frac{\frac{3}{4}}{2}\left(\frac{25}{16}x^2\right)+\dots \\ &= 1+\frac{5}{8}x+\frac{3}{8}\cdot\frac{25}{16}x^2+\dots \\ &= 1+\frac{5}{8}x+\frac{75}{128}x^2+\dots\end{aligned}$$

- **Step 3: Multiply by the external factor 1/2**

$$\begin{aligned}(4-5x)^{-1/2} &= \frac{1}{2}\left(1+\frac{5}{8}x+\frac{75}{128}x^2+\dots\right) \\ &= \frac{1}{2}+\frac{5}{16}x+\frac{75}{256}x^2+\dots\end{aligned}$$

2. Finding the value of k

The function is defined as $f(x)=(2+kx)(4-5x)^{-1/2}$. We are given that the **Maclaurin series** expansion is $1+\frac{3}{10}x+mx^2+\dots$.

- **Step 1: Set up the product**

$$\begin{aligned}f(x) &= (2+kx)\left(\frac{1}{2}+\frac{5}{16}x+\frac{75}{256}x^2+\dots\right) \\ &= 2\left(\frac{1}{2}+\frac{5}{16}x+\frac{75}{256}x^2\right)+kx\left(\frac{1}{2}+\frac{5}{16}x\right)+\dots \\ &= 1+\frac{5}{8}x+\frac{75}{128}x^2+\frac{k}{2}x+\frac{5k}{16}x^2+\dots\end{aligned}$$

- **Step 2: Compare the coefficient of x** The coefficient of x in the expansion is $\frac{5}{8}+\frac{k}{2}$. We equate this to the given coefficient $\frac{3}{10}$:

$$\begin{aligned}\frac{5}{8}+\frac{k}{2}&=\frac{3}{10}\\ \frac{k}{2}&=\frac{3}{10}-\frac{5}{8}\\ \frac{k}{2}&=\frac{12}{40}-\frac{25}{40}\\ \frac{k}{2}&=-\frac{13}{40}\\ k&=-\frac{13}{20}\end{aligned}$$

3. Finding the value of m

The constant m is the coefficient of the x^2 term in the expansion of $f(x)$.

- **Step 1: Identify the x^2 terms** From the product expansion in the previous step:

$$m=\frac{75}{128}+\frac{5k}{16}$$

- **Step 2: Substitute $k=-13/20$**

$$\begin{aligned}m&=\frac{75}{128}+\frac{5}{16}\left(-\frac{13}{20}\right)\\ &=\frac{75}{128}-\frac{13}{64}\\ &=\frac{75}{128}-\frac{26}{128}\\ &=\frac{49}{128}\end{aligned}$$

(a) $\frac{1}{2}+\frac{5}{16}x+\frac{75}{256}x^2$

(b) $\boxed{k=-\frac{13}{20}}$

(c) $\boxed{m=\frac{49}{128}}$

WMA11_P4(IAL)_Summer_2020_Q3

Solution

To find the volume of the solid of revolution formed by rotating the region R about the x-axis, we use the method of **disks**.

1. Determine the limits of integration The region R is bounded by the curve $y = e^{0.5x} - 2$, the y-axis ($x = 0$), and the x-axis ($y = 0$). We first find the x-intercept of the curve:

$$\begin{aligned} 0 &= e^{0.5x} - 2 \\ e^{0.5x} &= 2 \\ 0.5x &= \ln 2 \\ x &= 2\ln 2 \end{aligned}$$

Thus, the limits of integration are from $x = 0$ to $x = 2\ln 2$.

2. Set up the volume integral The **volume of revolution** V about the x-axis is given by the formula:

$$V = \pi \int_a^b y^2 \, dx$$

Substituting the given equation and limits:

$$V = \pi \int_0^{2\ln 2} (e^{0.5x} - 2)^2 \, dx$$

3. Expand and integrate First, expand the integrand:

$$(e^{0.5x} - 2)^2 = (e^{0.5x})^2 - 4e^{0.5x} + 4 = e^x - 4e^{0.5x} + 4$$

Now, perform the integration:

$$\begin{aligned} V &= \pi \int_0^{2\ln 2} (e^x - 4e^{0.5x} + 4) \, dx \\ &= \pi \left[e^x - \frac{4}{0.5} e^{0.5x} + 4x \right]_0^{2\ln 2} \\ &= \pi \left[e^x - 8e^{0.5x} + 4x \right]_0^{2\ln 2} \end{aligned}$$

4. Evaluate the definite integral Substitute the upper limit $x = 2\ln 2$:

- $e^{2\ln 2} = e^{\ln(2^2)} = 4$
- $8e^{0.5(2\ln 2)} = 8e^{\ln 2} = 8(2) = 16$
- $4(2\ln 2) = 8\ln 2$

Substitute the lower limit $x = 0$:

- $e^0 = 1$
- $8e^0 = 8$

- $4(0) = 0$

Calculate the difference:

$$\begin{aligned} V &= \pi((4 - 16 + 8\ln 2) - (1 - 8 + 0)) \\ &= \pi(-12 + 8\ln 2 - (-7)) \\ &= \pi(8\ln 2 - 5) \\ &= 8\pi\ln 2 - 5\pi \end{aligned}$$

The volume is expressed in the form $a\ln 2 + b$, where $a = 8\pi$ and $b = -5\pi$.

$\boxed{V = 8\pi\ln 2 - 5\pi}$

WMA11_P4(IAL)_Summer_2020_Q4

Solution

The curve is defined by the **parametric equations**:

$$x = 2t^2 - 6t, \quad y = t^3 - 4t, \quad t \in \mathbb{R}$$

1. Coordinates of A and B The curve intersects the x-axis when $y = 0$.

$$t^3 - 4t = 0$$
$$t(t^2 - 4) = 0$$
$$t(t-2)(t+2) = 0$$

The parameter values for the intercepts are $t = 0$, $t = 2$, and $t = -2$. We calculate the corresponding x coordinates:

- For $t = 0$: $x = 2(0)^2 - 6(0) = 0$. This corresponds to the origin $(0, 0)$.
- For $t = 2$: $x = 2(2)^2 - 6(2) = 8 - 12 = -4$. This corresponds to point $A(-4, 0)$.
- For $t = -2$: $x = 2(-2)^2 - 6(-2) = 8 + 12 = 20$. This corresponds to point $B(20, 0)$.

Thus, the coordinates of A are $(-4, 0)$ and B is $(20, 0)$.

2. Equation of the tangent at B To find the gradient of the tangent, we use the **chain rule** for derivatives:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

The derivatives with respect to t are:

$$\frac{dx}{dt} = 4t - 6, \quad \frac{dy}{dt} = 3t^2 - 4$$

At point B, $t = -2$:

$$\frac{dx}{dt}|_{t=-2} = 4(-2) - 6 = -14$$
$$\frac{dy}{dt}|_{t=-2} = 3(-2)^2 - 4 = 12 - 4 = 8$$
$$\frac{dy}{dx} = \frac{8}{-14} = -\frac{4}{7}$$

Using the point-slope form $y - y_1 = m(x - x_1)$ at $B(20, 0)$:

$$y - 0 = -\frac{4}{7}(x - 20)$$
$$7y = -4x + 80$$
$$7y + 4x - 80 = 0$$

3. x coordinate of point P Point P is where the tangent intersects the curve again. We substitute the parametric expressions for x and y into the tangent equation:

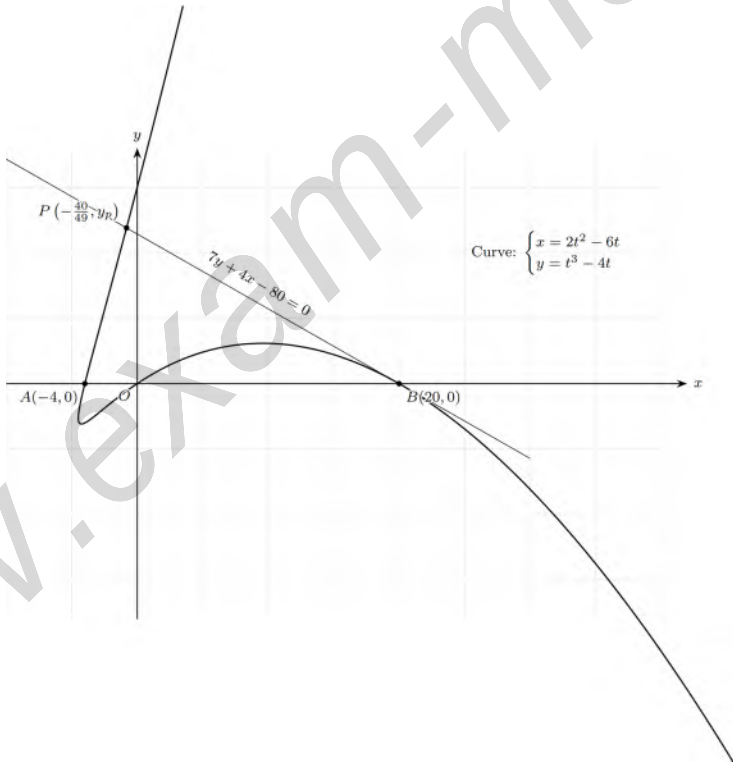
$$7(t^3 - 4t) + 4(2t^2 - 6t) - 80 = 0$$
$$7t^3 - 28t + 8t^2 - 24t - 80 = 0$$
$$7t^3 + 8t^2 - 52t - 80 = 0$$

Since the line is tangent to the curve at B (where $t = -2$), $(t + 2)$ must be a repeated factor of the polynomial. We can perform **polynomial division** or use **synthetic division** by $(t + 2)^2 = t^2 + 4t + 4$:

$$7t^3 + 8t^2 - 52t - 80 = (t + 2)^2(7t - 20) = 0$$

The intersection points occur at $t = -2$ (point B) and $t = \frac{20}{7}$ (point P). To find the x coordinate of P:

$$x_P = 2\left(\frac{20}{7}\right)^2 - 6\left(\frac{20}{7}\right)$$
$$= 2\left(\frac{400}{49}\right) - \frac{120}{7}$$
$$= \frac{800}{49} - \frac{840}{49}$$
$$= -\frac{40}{49}$$



The x coordinate of P is $\boxed{-\frac{40}{49}}$.

WMA11_P4(IAL)_Summer_2020_Q5

Solution

1. Indefinite Integration of $\frac{\ln x}{x^2}$

To find the integral $I = \int \frac{\ln x}{x^2}\,dx$, we apply the method of **Integration by Parts**. The formula for integration by parts is:

$$\int u\,dv = uv - \int v\,du$$

We choose the components based on the **LIATE rule**:

- Let $u = \ln x$, then $du = \frac{1}{x}\,dx$.
- Let $dv = \frac{1}{x^2}\,dx = x^{-2}\,dx$, then $v = \int x^{-2}\,dx = -x^{-1} = -\frac{1}{x}$.

Substituting these into the formula:

$$\begin{aligned}\int \frac{\ln x}{x^2}\,dx &= (\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx \\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C\end{aligned}$$

Factoring out $-\frac{1}{x}$:

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}(1 + \ln x) + C$$

2. Exact Area of Region R

The region R is bounded by the curve $y = \frac{3+2x+\ln x}{x^2}$, the x-axis, and the vertical lines $x = 2$ and $x = 4$. The area A is given by the **definite integral**:

$$A = \int_2^4 \frac{3 + 2x + \ln x}{x^2}\,dx$$

We decompose the integrand into simpler terms:

$$\begin{aligned}A &= \int_2^4 \left(\frac{3}{x^2} + \frac{2x}{x^2} + \frac{\ln x}{x^2}\right)dx \\ &= \int_2^4 \left(3x^{-2} + \frac{2}{x} + \frac{\ln x}{x^2}\right)dx\end{aligned}$$

Using the result from part (a), the **antiderivative** is:

$$F(x) = -\frac{3}{x} + 2\ln x - \frac{1}{x}(1 + \ln x)$$

Combining the terms involving $1/x$:

$$\begin{aligned}F(x) &= 2\ln x - \frac{3}{x} - \frac{1}{x} - \frac{\ln x}{x} \\ &= 2\ln x - \frac{4}{x} - \frac{\ln x}{x}\end{aligned}$$

Now, evaluate the definite integral from $x = 2$ to $x = 4$:

$$\begin{aligned}A &= \left[2\ln x - \frac{4+\ln x}{x}\right]_2^4 \\ &= \left(2\ln 4 - \frac{4+\ln 4}{4}\right) - \left(2\ln 2 - \frac{4+\ln 2}{2}\right)\end{aligned}$$

Using the **logarithm power rule** $\ln 4 = \ln(2^2) = 2\ln 2$:

$$\begin{aligned}A &= \left(4\ln 2 - \frac{4+2\ln 2}{4}\right) - \left(2\ln 2 - \frac{4+\ln 2}{2}\right) \\ &= \left(4\ln 2 - 1 - \frac{1}{2}\ln 2\right) - \left(2\ln 2 - 2 - \frac{1}{2}\ln 2\right) \\ &= \left(\frac{7}{2}\ln 2 - 1\right) - \left(\frac{3}{2}\ln 2 - 2\right) \\ &= \frac{7}{2}\ln 2 - \frac{3}{2}\ln 2 - 1 + 2 \\ &= 2\ln 2 + 1\end{aligned}$$

Alternatively, this can be written as $\ln 4 + 1$.

(a) $\boxed{-\frac{1}{x}(1+\ln x) + C}$

(b) $\boxed{1 + 2\ln 2}$

WMA11_P4(IAL)_Summer_2020_Q6

Solution

1. Logarithmic Differentiation

To find the derivative of the function $y = x^{\sin x}$ where $x > 0$ and $y > 0$, we employ **logarithmic differentiation**.

- **Step 1: Take the natural logarithm of both sides.** Using the property $\ln(a^b) = b\ln a$:

$$\ln y = \ln(x^{\sin x})$$

$$\ln y = \sin x \ln x$$

- **Step 2: Differentiate implicitly with respect to x.** We apply the **chain rule** to the left side and the **product rule** to the right side:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{d}{dx}\sin x\right)\ln x + \sin x\left(\frac{d}{dx}\ln x\right)$$
$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \sin x\left(\frac{1}{x}\right)$$

- **Step 3: Solve for $\frac{dy}{dx}$.** Multiply both sides by y:

$$\frac{dy}{dx} = y\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

$$\boxed{\frac{dy}{dx} = y\left(\cos x \ln x + \frac{\sin x}{x}\right)}$$

2. Stationary Points

A **stationary point** occurs where the first derivative of the function is zero, i.e., $\frac{dy}{dx} = 0$.

- **Step 1: Set the derivative expression to zero.**

$$y\left(\cos x \ln x + \frac{\sin x}{x}\right) = 0$$

- **Step 2: Analyze the factors.** Since the problem states $y > 0$, the factor y can never be zero. Therefore, the term inside the parentheses must vanish:

$$\cos x \ln x + \frac{\sin x}{x} = 0$$

- **Step 3: Rearrange to the required form.** Multiply the entire equation by $\frac{x}{\cos x}$ (assuming $\cos x \neq 0$):

$$x \ln x + \frac{\sin x}{\cos x} = 0$$
$$x \ln x + \tan x = 0$$

Rearranging the terms gives:

$$\tan x + x \ln x = 0$$

This confirms that the x-coordinates of the stationary points of curve C are indeed solutions to the equation $\tan x + x \ln x = 0$.

WMA11_P4(IAL)_Summer_2020_Q7

Solution

1. Definite Integration via Substitution

To evaluate the integral $I = \int_1^5 \frac{3x}{\sqrt{2x-1}}\,dx$, we apply the **u-substitution** method.

- **Step 1: Define the substitution** Let $u = \sqrt{2x-1}$. Then:

$$u^2 = 2x - 1$$
$$x = \frac{u^2+1}{2}$$

  Differentiating x with respect to u:

$$dx = u\,du$$

- **Step 2: Change the limits of integration**
  - When $x = 1$, $u = \sqrt{2(1)-1} = 1$.
  - When $x = 5$, $u = \sqrt{2(5)-1} = 3$.
- **Step 3: Substitute and simplify**

$$I = \int_1^3 \frac{3\left(\frac{u^2+1}{2}\right)}{u}(u\,du)$$
$$= \frac{3}{2}\int_1^3 (u^2+1)\,du$$

- **Step 4: Integrate and evaluate**

$$I = \frac{3}{2}\left[\frac{u^3}{3} + u\right]_1^3$$
$$= \frac{3}{2}\left(\left(\frac{3^3}{3} + 3\right) - \left(\frac{1^3}{3} + 1\right)\right)$$
$$= \frac{3}{2}\left((9+3) - \left(\frac{1}{3} + 1\right)\right)$$
$$= \frac{3}{2}\left(12 - \frac{4}{3}\right)$$
$$= \frac{3}{2}\left(\frac{32}{3}\right)$$
$$= 16$$

The value of the integral is $\boxed{16}$.

2. Indefinite Integration via Partial Fractions

To find the integral $J = \int \frac{6x^2-16}{(x+1)(2x-3)}\,dx$, we first observe that the degree of the numerator is equal to the degree of the denominator (2). We must perform **polynomial long division** or algebraic manipulation first.

- **Step 1: Expand the denominator**

$$(x+1)(2x-3) = 2x^2 - x - 3$$

- **Step 2: Perform division** We can write the integrand as:

$$\frac{6x^2-16}{2x^2-x-3} = 3 + \frac{3x-7}{(x+1)(2x-3)}$$

- **Step 3: Partial Fraction Decomposition** Let $\frac{3x-7}{(x+1)(2x-3)} = \frac{A}{x+1} + \frac{B}{2x-3}$. Multiplying by the common denominator:

$$3x - 7 = A(2x-3) + B(x+1)$$

  - Set $x = -1$: $-10 = A(-5) \implies A = 2$.
  - Set $x = 1.5$: $-2.5 = B(2.5) \implies B = -1$. Thus:

$$\frac{6x^2-16}{(x+1)(2x-3)} = 3 + \frac{2}{x+1} - \frac{1}{2x-3}$$

- **Step 4: Integrate**

$$J = \int \left(3 + \frac{2}{x+1} - \frac{1}{2x-3}\right) dx$$
$$= 3x + 2\ln|\ x+1\ | - \frac{1}{2}\ln|\ 2x-3\ | + C$$

The indefinite integral is $\boxed{3x + 2\ln|\ x+1\ | - \frac{1}{2}\ln|\ 2x-3\ | + C}$.

WMA11_P4(IAL)_Summer_2020_Q8

Solution

The problem involves finding the intersection of two lines in three-dimensional space and determining the coordinates of a specific point based on a perpendicularity condition.

1. Finding the position vector of the intersection point X

The equations for the lines l_1 and l_2 are given by:

$$l_1: \mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

At the intersection point X, the position vectors are equal. We equate the components:

$$4 + 3\lambda = 2 + 2\mu \quad (1)$$
$$-3 - 2\lambda = -\mu \quad (2)$$
$$2 - \lambda = -9 - 3\mu \quad (3)$$

From equation (2), we can express μ in terms of λ:

$$\mu = 3 + 2\lambda$$

Substitute this into equation (1):

$$4 + 3\lambda = 2 + 2(3 + 2\lambda)$$
$$4 + 3\lambda = 2 + 6 + 4\lambda$$
$$4 + 3\lambda = 8 + 4\lambda$$
$$\lambda = -4$$

Now, find μ:

$$\mu = 3 + 2(-4) = -5$$

Verify with equation (3):

$$\text{LHS} = 2 - (-4) = 6$$
$$\text{RHS} = -9 - 3(-5) = -9 + 15 = 6$$

Since LHS = RHS, the lines intersect at $\lambda = -4$. The position vector $\overrightarrow{OX}$ is:

$$\overrightarrow{OX} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} - 4\begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 - 12 \\ -3 + 8 \\ 2 + 4 \end{pmatrix} = \begin{pmatrix} -8 \\ 5 \\ 6 \end{pmatrix}$$

2. Calculating the coordinates of point Q

Point P is $(10, -7, 0)$. Point Q lies on l_2, so its position vector $\overrightarrow{OQ}$ is:

$$\overrightarrow{OQ} = \begin{pmatrix} 2 + 2\mu \\ -\mu \\ -9 - 3\mu \end{pmatrix}$$

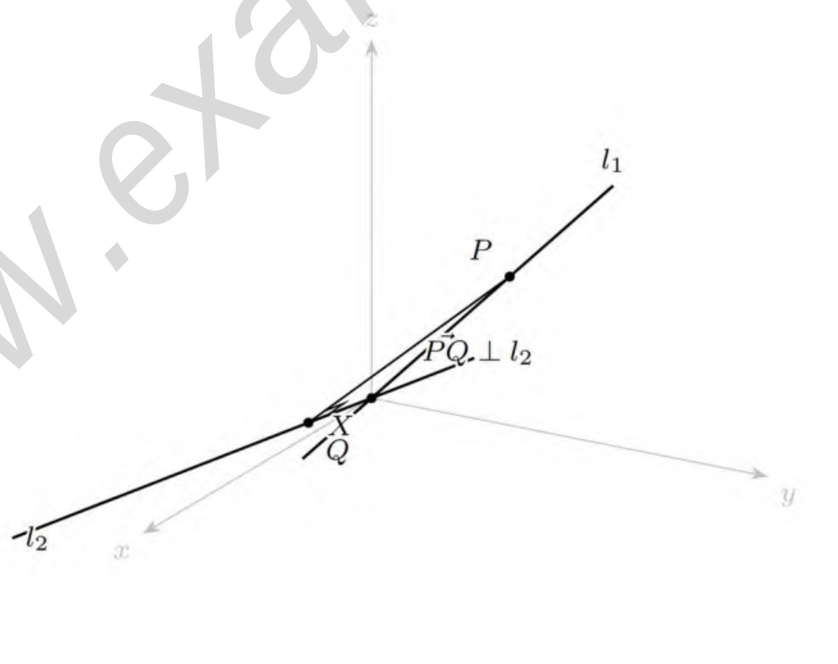
The vector $\overrightarrow{PQ}$ is given by:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{pmatrix} 2+2\mu \\ -\mu \\ -9-3\mu \end{pmatrix} - \begin{pmatrix} 10 \\ -7 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\mu-8 \\ 7-\mu \\ -3\mu-9 \end{pmatrix}$$

The vector $\overrightarrow{PQ}$ is perpendicular to l_2. The direction vector of l_2 is $\mathbf{d}_2 = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$. Using the **dot product** property for perpendicular vectors:

$$\overrightarrow{PQ} \cdot \mathbf{d}_2 = 0$$

$$\begin{pmatrix} 2\mu-8 \\ 7-\mu \\ -3\mu-9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = 0$$
$$2(2\mu-8) - 1(7-\mu) - 3(-3\mu-9) = 0$$
$$4\mu - 16 - 7 + \mu + 9\mu + 27 = 0$$
$$14\mu + 4 = 0$$
$$\mu = -\frac{4}{14} = -\frac{2}{7}$$



Substitute $\mu = -2/7$ back into the expression for $\overrightarrow{OQ}$:

$$x_Q = 2 + 2\left(-\frac{2}{7}\right) = 2 - \frac{4}{7} = \frac{10}{7}$$
$$y_Q = -\left(-\frac{2}{7}\right) = \frac{2}{7}$$
$$z_Q = -9 - 3\left(-\frac{2}{7}\right) = -9 + \frac{6}{7} = -\frac{57}{7}$$

The coordinates of Q are $\left(\frac{10}{7}, \frac{2}{7}, -\frac{57}{7}\right)$.

(a) Position vector of X: $\overrightarrow{\mathbf{OX}} = \begin{pmatrix} \mathbf{-8} \\ \mathbf{5} \\ \mathbf{6} \end{pmatrix}$

(b) Coordinates of Q: $\left(\frac{\mathbf{10}}{\mathbf{7}}, \frac{\mathbf{2}}{\mathbf{7}}, -\frac{\mathbf{57}}{\mathbf{7}}\right)$

WMA11_P4(IAL)_Summer_2020_Q9

Solution

1. Separation of Variables

The growth of the bacteria is modeled by the **differential equation**:

$$\frac{dA}{dt} = \frac{A^{3/2}}{5t^2}, \quad t > 0$$

To solve this, we use the method of **separation of variables**. We rearrange the equation to group all terms involving A on the left and all terms involving t on the right:

$$A^{-3/2}\,dA = \frac{1}{5}t^{-2}\,dt$$

2. Integration

Integrating both sides of the equation:

$$\int A^{-3/2}\,dA = \int \frac{1}{5}t^{-2}\,dt$$

- The left-hand side integral is:

$$\int A^{-3/2}\,dA = \frac{A^{-1/2}}{-1/2} = -2A^{-1/2}$$

- The right-hand side integral is:

$$\int \frac{1}{5}t^{-2}\,dt = \frac{1}{5}\left(\frac{t^{-1}}{-1}\right) + C = -\frac{1}{5t} + C$$

Equating the two results:

$$-2A^{-1/2} = -\frac{1}{5t} + C$$

Multiplying through by -1:

$$\frac{2}{\sqrt{A}} = \frac{1}{5t} + K \quad (\text{where} \quad K = -C)$$

3. Applying Initial Conditions

We are given that $A = 2.25$ when $t = 3$. Substituting these values to find the **constant of integration** K:

$$\frac{2}{\sqrt{2.25}} = \frac{1}{5(3)} + K$$
$$\frac{2}{1.5} = \frac{1}{15} + K$$
$$\frac{4}{3} = \frac{1}{15} + K$$
$$K = \frac{20}{15} - \frac{1}{15}$$
$$K = \frac{19}{15}$$

4. Expressing A in the Required Form

Substitute K back into the equation:

$$\frac{2}{\sqrt{A}} = \frac{1}{5t} + \frac{19}{15}$$

To combine the terms on the right, find a common denominator ($15t$):

$$\frac{2}{\sqrt{A}} = \frac{3}{15t} + \frac{19t}{15t}$$
$$\frac{2}{\sqrt{A}} = \frac{3 + 19t}{15t}$$

Invert both sides:

$$\frac{\sqrt{A}}{2} = \frac{15t}{19t + 3}$$

Multiply by 2:

$$\sqrt{A} = \frac{30t}{19t + 3}$$

Squaring both sides gives the required form:

$$A = \left(\frac{30t}{19t + 3}\right)^2$$

Comparing this to the form $A = \left(\frac{pt}{qt+r}\right)^2$, we identify the integers:

$$\boxed{p = 30, \quad q = 19, \quad r = 3}$$

5. Finding the Limiting Area

To find the limit of the area as $t \to \infty$, we evaluate the **horizontal asymptote** of the function $A(t)$:

$$\lim_{t\to\infty} A = \lim_{t\to\infty} \left(\frac{30t}{19t + 3}\right)^2$$
$$= \lim_{t\to\infty} \left(\frac{30}{19 + \frac{3}{t}}\right)^2$$

As $t \to \infty$, the term $\frac{3}{t} \to 0$:

$$\text{Limit} = \left(\frac{30}{19}\right)^2$$
$$= \frac{900}{361}$$

Calculating the decimal value:

$$\frac{900}{361} \approx 2.49307...$$

The limiting area is:

$$\boxed{\frac{900}{361} \approx 2.49\,\text{cm}^2}$$