WMA11_P3(IAL)_Summer_2020_Q1

Solution

1. Transformation to a Quadratic Equation

To solve the equation $2\cos 2x = 7\cos x$ for $0^\circ \leq x < 360^\circ$, we first apply the **double angle formula** for cosine:

$$\cos 2x = 2\cos^2 x - 1$$

Substituting this into the original equation:

$$2(2\cos^2 x - 1) = 7\cos x$$
$$4\cos^2 x - 2 = 7\cos x$$
$$4\cos^2 x - 7\cos x - 2 = 0$$

2. Solving the Quadratic Equation

Let $u = \cos x$. The equation becomes a quadratic in terms of u:

$$4u^2 - 7u - 2 = 0$$

We can solve this using the **quadratic formula** $u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$u = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(4)(-2)}}{2(4)}$$
$$= \frac{7 \pm \sqrt{49 + 32}}{8}$$
$$= \frac{7 \pm \sqrt{81}}{8}$$
$$= \frac{7 \pm 9}{8}$$

This gives two possible values for u:

- $u_1 = \frac{7+9}{8} = \frac{16}{8} = 2$
- $u_2 = \frac{7-9}{8} = \frac{-2}{8} = -0.25$

3. Finding the Values of x

We now solve for x using the relationship $\cos x = u$:

- **Case 1:** $\cos x = 2$ Since the range of the cosine function is $-1 \leq \cos x \leq 1$, there are no real solutions for x in this case.
- **Case 2:** $\cos x = -0.25$ We find the principal value using the inverse cosine function:

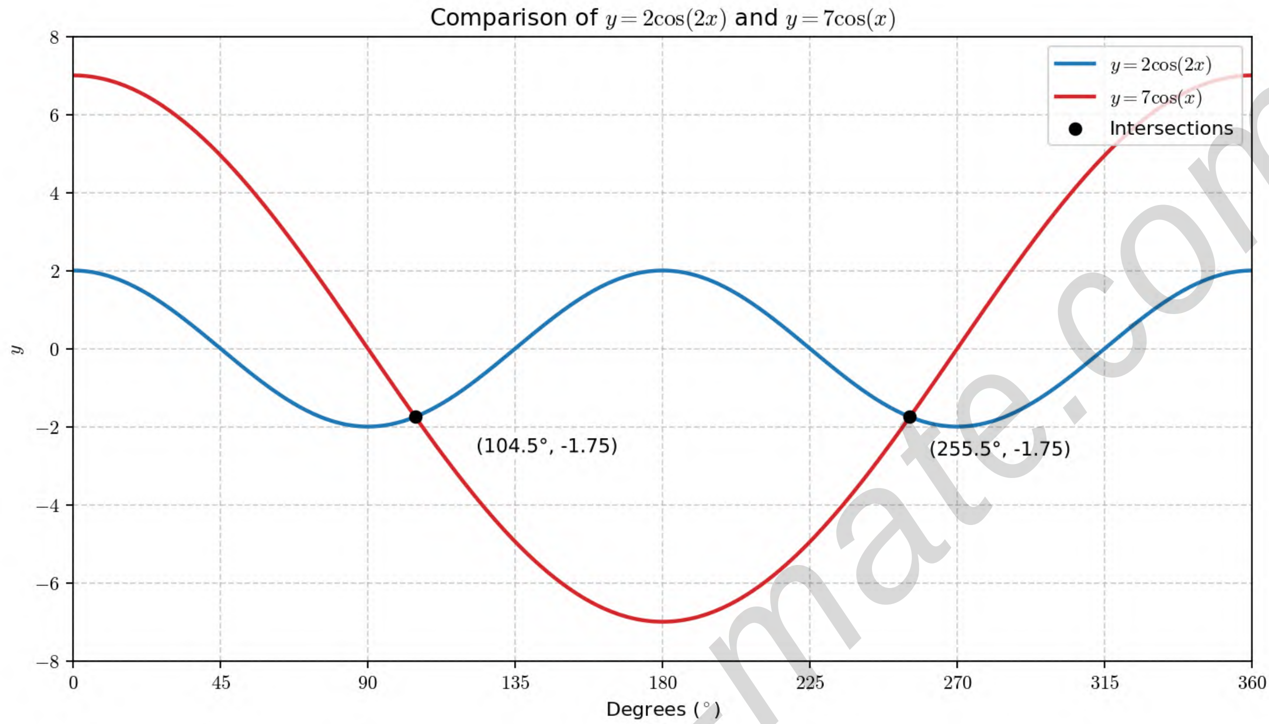
  $$x_1 = \arccos(-0.25) \approx 104.4775^\circ$$

  Rounding to one decimal place, we get $x_1 \approx 104.5^\circ$.

  Since the cosine function is symmetric about the x-axis in the **unit circle**, the second solution in the interval $0^\circ \leq x < 360^\circ$ is:

$$\begin{aligned} x_2 &= 360^\circ - x_1 \\ &= 360^\circ - 104.4775^\circ \\ &= 255.5225^\circ \end{aligned}$$

Rounding to one decimal place, we get $x_2 \approx 255.5^\circ$.



4. Final Solutions

The solutions within the specified range, rounded to one decimal place, are:

$$\boxed{104.5^\circ, 255.5^\circ}$$

WMA11_P3(IAL)_Summer_2020_Q2

Solution

1. Conversion to Exponential Form

To convert the logarithmic equation into an exponential form, we apply the definition of a **logarithm**. Given the equation:

$$\log_{10} N = 0.0646t + 1.478$$

We raise 10 to the power of both sides of the equation:

$$\begin{aligned} 10^{\log_{10} N} &= 10^{0.0646t+1.478} \\ N &= 10^{1.478} \cdot 10^{0.0646t} \\ N &= 10^{1.478} \cdot (10^{0.0646})^t \end{aligned}$$

This matches the required form $N = ab^t$, where:

- $a = 10^{1.478}$
- $b = 10^{0.0646}$

Calculating the constants:

- For a:

  $$a = 10^{1.478} \approx 30.06076$$

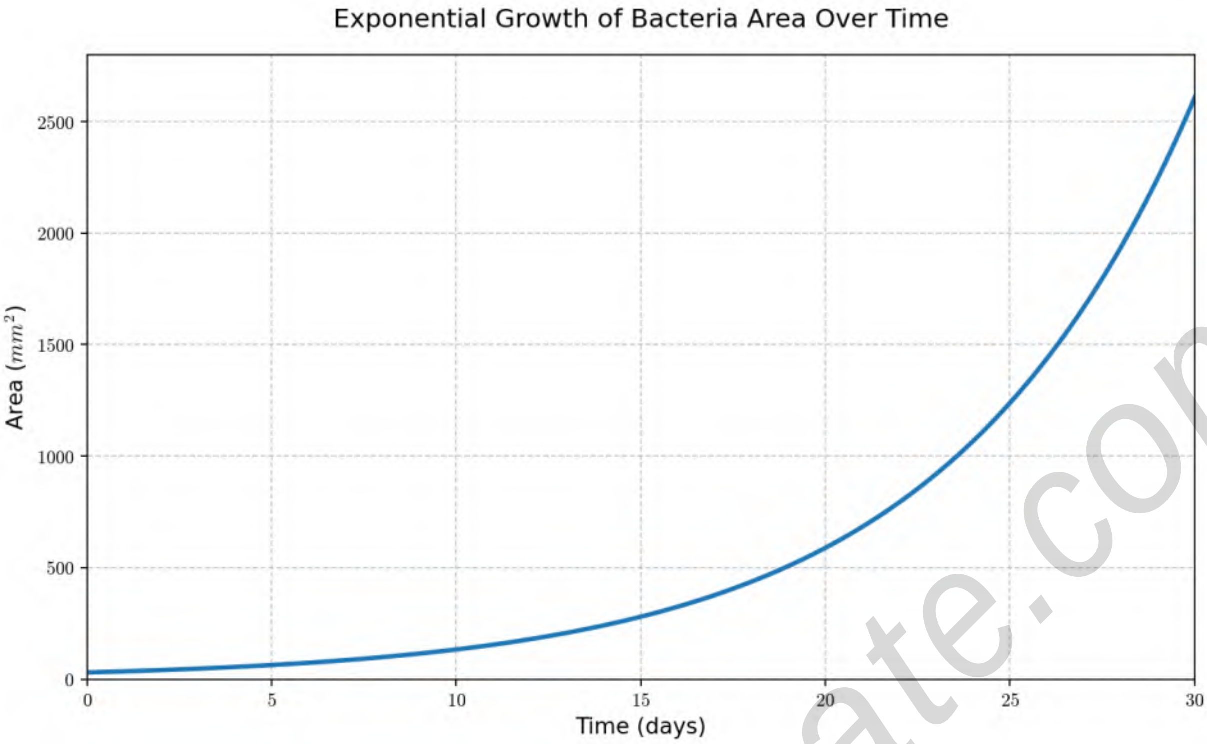
  Rounding to the nearest integer, we get $a = 30$.
- For b:

  $$b = 10^{0.0646} \approx 1.16037$$

  Rounding to 3 significant figures, we get $b = 1.16$.

Thus, the equation in the form $N = ab^t$ is:

$$\boxed{N = 30 \cdot 1.16^t, \text{ where } \; a = 30, b = 1.16}$$



2. Finding the Area After 30 Days

To find the area N when $t = 30$, we substitute $t = 30$ into the original logarithmic model to maintain maximum precision before rounding:

$$\begin{aligned}\log_{10} N &= 0.0646(30) + 1.478 \\ &= 1.938 + 1.478 \\ &= 3.416\end{aligned}$$

Now, solve for N:

$$\begin{aligned}N &= 10^{3.416} \\ &\approx 2606.1537\end{aligned}$$

Rounding to 2 significant figures:

- The first two significant digits are 2 and 6.
- The third digit is 0, so we do not round up.
- $N \approx 2600\,\text{mm}^2$.

$$\boxed{2600\,\text{mm}^2}$$

WMA11_P3(IAL)_Summer_2020_Q3

Solution

Given the function $f(x) = \frac{2x+3}{\sqrt{4x-1}}$ for $x > \frac{1}{4}$, we proceed with the following steps to find its derivative and range.

1. Finding the derivative $f'(x)$

To differentiate $f(x)$, we apply the **quotient rule**, which states that for $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. Let:

- $u = 2x + 3 \implies \frac{du}{dx} = 2$
- $v = (4x-1)^{1/2} \implies \frac{dv}{dx} = \frac{1}{2}(4x-1)^{-1/2} \cdot 4 = 2(4x-1)^{-1/2}$

Substituting these into the quotient rule formula:

$$f'(x) = \frac{(4x-1)^{1/2} \cdot 2 - (2x+3) \cdot 2(4x-1)^{-1/2}}{(4x-1)}$$
$$= \frac{2(4x-1)^{1/2} - \frac{2(2x+3)}{(4x-1)^{1/2}}}{4x-1}$$

To simplify, we multiply the numerator and denominator by $(4x-1)^{1/2}$:

$$f'(x) = \frac{2(4x-1) - 2(2x+3)}{(4x-1)^{3/2}}$$
$$= \frac{8x - 2 - 4x - 6}{(4x-1)^{3/2}}$$
$$= \frac{4x-8}{(4x-1)^{3/2}}$$

Factoring the numerator gives the simplest form:

$$\boxed{f'(x) = \frac{4(x-2)}{(4x-1)^{3/2}}}$$

2. Finding the range of f

To determine the range, we analyze the behavior of the function based on its derivative and limits.

- **Stationary Point:** Set $f'(x) = 0$.

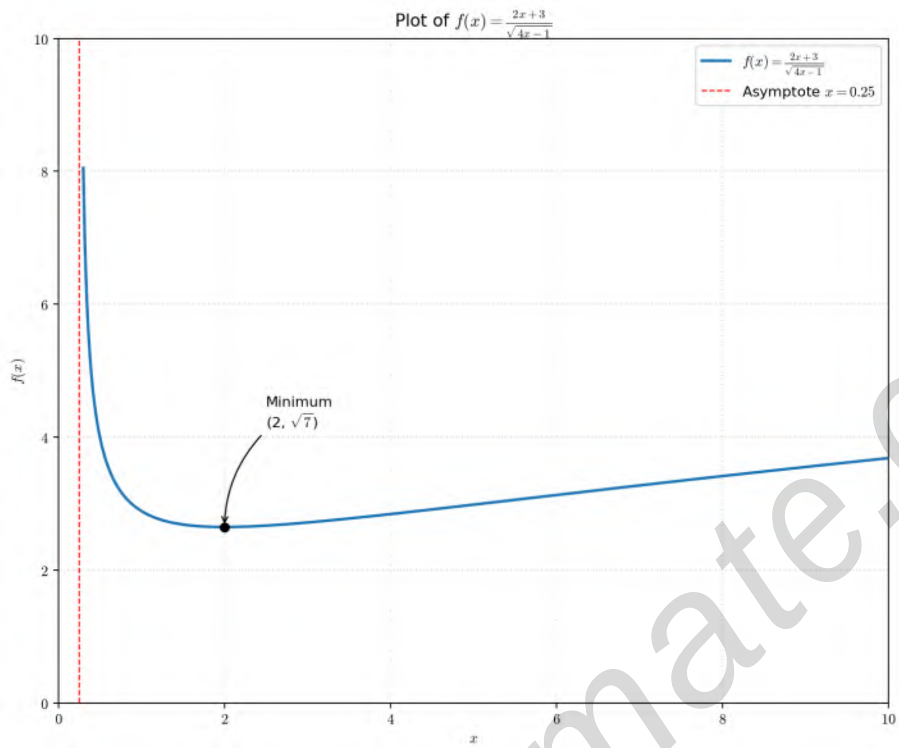
$$4(x-2) = 0 \implies x = 2$$

  Since $x = 2$ is within the domain $x > \frac{1}{4}$, we calculate the corresponding y-value:

$$f(2) = \frac{2(2)+3}{\sqrt{4(2)-1}} = \frac{7}{\sqrt{7}} = \sqrt{7}$$

- **End Behavior:**

- As $x \to \frac{1}{4^+}$, the denominator $\sqrt{4x-1} \to 0^+$, so $f(x) \to \infty$.
- As $x \to \infty$, we examine the leading terms: $f(x) \approx \frac{2x}{\sqrt{4x}} = \frac{2x}{2\sqrt{x}} = \sqrt{x}$. Thus, $f(x) \to \infty$.



Since the function is continuous on its domain, starts at ∞, decreases to a minimum of $\sqrt{7}$ at $x = 2$, and then increases back to ∞, the **range** is the set of all values greater than or equal to the minimum value.

$$\boxed{f(x) \geq \sqrt{7}}$$

WMA11_P3(IAL)_Summer_2020_Q4

Solution

The function is defined as $f(x) = 21 - 2 \mid 2 - x \mid$ for $x \geq 0$.

1. Evaluation of $ff(6)$

- First, evaluate $f(6)$:

$$\begin{aligned} f(6) &= 21 - 2 \mid 2 - 6 \mid \\ &= 21 - 2 \mid -4 \mid \\ &= 21 - 2(4) \\ &= 21 - 8 \\ &= 13 \end{aligned}$$

- Next, evaluate $f(f(6)) = f(13)$:

$$\begin{aligned} f(13) &= 21 - 2 \mid 2 - 13 \mid \\ &= 21 - 2 \mid -11 \mid \\ &= 21 - 2(11) \\ &= 21 - 22 \\ &= -1 \end{aligned}$$

$$\boxed{ff(6) = -1}$$

2. Solving the equation $f(x) = 5x$

- We consider the two cases for the **absolute value** expression $\mid 2 - x \mid$:
  - **Case 1:** $0 \leq x \leq 2$. Here, $2 - x \geq 0$, so $\mid 2 - x \mid = 2 - x$.

$$\begin{aligned} 21 - 2(2 - x) &= 5x \\ 21 - 4 + 2x &= 5x \\ 17 &= 3x \\ x &= \frac{17}{3} \approx 5.67 \end{aligned}$$

    Since 5.67 is not in the range $0 \leq x \leq 2$, this is not a valid solution for this branch.
  - **Case 2:** $x > 2$. Here, $2 - x < 0$, so $\mid 2 - x \mid = -(2 - x) = x - 2$.

$$\begin{aligned} 21 - 2(x - 2) &= 5x \\ 21 - 2x + 4 &= 5x \\ 25 &= 7x \\ x &= \frac{25}{7} \approx 3.57 \end{aligned}$$

    Since $3.57 > 2$, this is a valid solution.

$$\boxed{x = \frac{25}{7}}$$

3. Range of k for exactly two roots

- The function $f(x) = 21 - 2 \mid 2 - x \mid$ is a V-shaped graph opening downwards.
- The **vertex** occurs when $2 - x = 0$, which is at $x = 2$.
- The y-coordinate of the vertex is $f(2) = 21 - 2 \mid 0 \mid = 21$.
- At the boundary $x = 0$, the value is $f(0) = 21 - 2 \mid 2 \mid = 17$.
- For $x > 2$, the function decreases indefinitely: $f(x) = 21 - 2(x - 2) = 25 - 2x$.
- To have exactly two roots for $f(x) = k$, the horizontal line $y = k$ must intersect the graph twice. Looking at the sketch:
  - The peak is at $(2, 21)$.
  - The left endpoint of the domain is at $(0, 17)$.
  - Between $y = 17$ and $y = 21$, any horizontal line hits both the left branch ($0 \leq x \leq 2$) and the right branch ($x > 2$).
  - At $k = 21$, there is only one root (the vertex).
  - At $k = 17$, there are two roots ($x = 0$ and another on the right branch).
  - Below $k = 17$, the line only hits the right branch because the left branch terminates at $x = 0$.
- Thus, the condition for two roots is $17 \leq k < 21$.

$$\boxed{17 \leq k < 21}$$

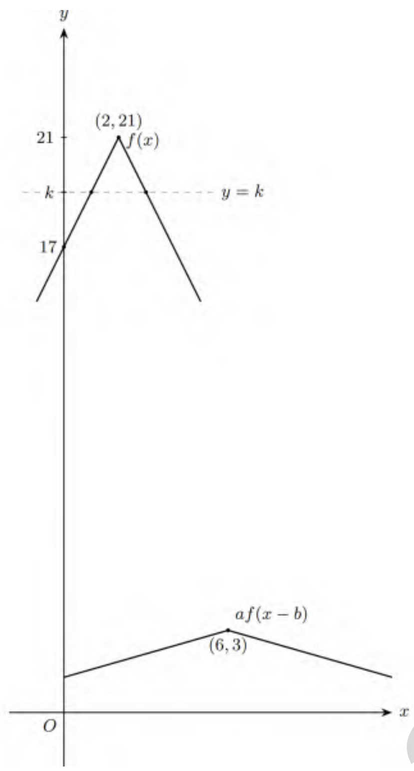
4. Transformations and Constants a and b

- The original vertex is at $(2, 21)$.
- The transformed function is $y = af(x - b)$.
- A transformation of the form $f(x - b)$ represents a **horizontal translation** by b units.
- A transformation of the form $af(x)$ represents a **vertical stretch** by a factor of a.
- The new vertex is given as $(6, 3)$.
- **Finding b:** The x-coordinate moves from 2 to 6.

$$\begin{aligned} 2 + b &= 6 \\ b &= 4 \end{aligned}$$

- **Finding a:** The y-coordinate moves from 21 to 3.

$$\begin{aligned} a \cdot 21 &= 3 \\ a &= \frac{3}{21} \\ a &= \frac{1}{7} \end{aligned}$$



$$\boxed{a = \frac{1}{7}, \quad b = 4}$$

WMA11_P3(IAL)_Summer_2020_Q5

Solution

1. Derivation of the Triple-Angle Identity

To show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$, we utilize the **angle addition formula** for sine:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

- Let $A = 2x$ and $B = x$:

$$\begin{aligned}\sin 3x &= \sin(2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x\end{aligned}$$

- Substitute the **double-angle formulas** $\sin 2x = 2\sin x \cos x$ and $\cos 2x = 1 - 2\sin^2 x$:

$$\begin{aligned}\sin 3x &= (2\sin x \cos x)\cos x + (1 - 2\sin^2 x)\sin x \\ &= 2\sin x \cos^2 x + \sin x - 2\sin^3 x\end{aligned}$$

- Apply the **Pythagorean identity** $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned}\sin 3x &= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x \\ &= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

  Thus, the identity is proven.

2. Evaluation of the Definite Integral

To evaluate $I = \int_0^{\frac{\pi}{3}} \sin^3 x \, dx$, we first rearrange the identity derived in part (a) to isolate $\sin^3 x$:

$$\begin{aligned}4\sin^3 x &= 3\sin x - \sin 3x \\ \sin^3 x &= \frac{1}{4}(3\sin x - \sin 3x)\end{aligned}$$

- Substitute this expression into the integral:

$$\begin{aligned}I &= \int_0^{\frac{\pi}{3}} \frac{1}{4}(3\sin x - \sin 3x)\, dx \\ &= \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{3}}\end{aligned}$$

- Evaluate at the upper limit $x = \frac{\pi}{3}$:

$$\begin{aligned}\text{Upper} &= -3\cos\left(\frac{\pi}{3}\right) + \frac{1}{3}\cos\left(3 \cdot \frac{\pi}{3}\right) \\ &= -3\left(\frac{1}{2}\right) + \frac{1}{3}\cos(\pi) \\ &= -\frac{3}{2} + \frac{1}{3}(-1) = -\frac{3}{2} - \frac{1}{3} = -\frac{11}{6}\end{aligned}$$

- Evaluate at the lower limit $x = 0$:

$$\text{Lower} = -3\cos(0) + \frac{1}{3}\cos(0)$$
$$= -3(1) + \frac{1}{3}(1) = -\frac{9}{3} + \frac{1}{3} = -\frac{8}{3}$$

- Calculate the difference:

$$I = \frac{1}{4}\left(-\frac{11}{6} - \left(-\frac{8}{3}\right)\right)$$
$$= \frac{1}{4}\left(-\frac{11}{6} + \frac{16}{6}\right)$$
$$= \frac{1}{4}\left(\frac{5}{6}\right) = \frac{5}{24}$$

$$\boxed{\frac{5}{24}}$$

WMA11_P3(IAL)_Summer_2020_Q6

Solution

The problem involves analyzing two curves, C_1 and C_2, defined by the following equations:

$$C_1 : y = 5e^{x-1} + 3$$
$$C_2 : y = 10 - x^2$$

1. Finding the x-coordinate of point P Point P lies on C_1 and has a y-coordinate of 18. We substitute $y = 18$ into the equation for C_1:

$$18 = 5e^{x-1} + 3$$
$$15 = 5e^{x-1}$$
$$3 = e^{x-1}$$

Taking the **natural logarithm** of both sides:

$$\ln(3) = x - 1$$
$$x = 1 + \ln(3)$$
$$x = \ln(e) + \ln(3)$$
$$x = \ln(3e)$$

Comparing this to the required form $\ln k$, we find $k = 3e$. $\boxed{x = \ln(3e)}$

2. Showing that $\alpha = 1.134$ to 3 decimal places The curves intersect where $5e^{x-1} + 3 = 10 - x^2$. We define a function $f(x)$ to use the **Intermediate Value Theorem**:

$$f(x) = 5e^{x-1} + x^2 - 7$$

To show that the root α is 1.134 to 3 decimal places, we test the boundaries of the interval $[1.1335, 1.1345]$:

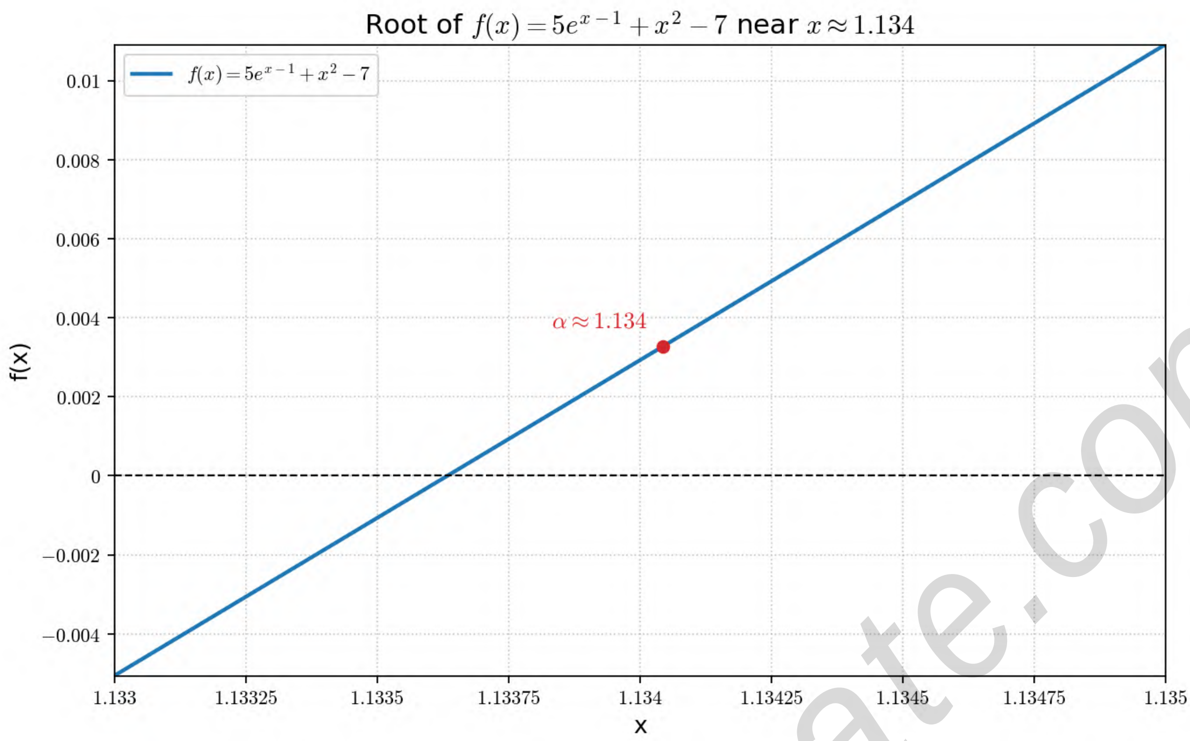
- For $x = 1.1335$:

$$f(1.1335) = 5e^{1.1335-1} + (1.1335)^2 - 7 \approx -0.000708$$

- For $x = 1.1345$:

$$f(1.1345) = 5e^{1.1345-1} + (1.1345)^2 - 7 \approx 0.005161$$

Since there is a sign change ($f(1.1335) < 0$ and $f(1.1345) > 0$) and the function is continuous, a root α exists in the interval $(1.1335, 1.1345)$. Therefore, $\alpha = 1.134$ to 3 decimal places.



3. Iterative approximation for β We use the **fixed-point iteration** formula:

$$x_{n+1} = -\sqrt{7 - 5e^{x_n - 1}}$$

Starting with $x_1 = -3$:

- Calculation for x_2:

$$\begin{aligned} x_2 &= -\sqrt{7 - 5e^{-3-1}} \\ &= -\sqrt{7 - 5e^{-4}} \\ &\approx -2.628345 \end{aligned}$$

- Continuing the iteration to find β (the value where x_n converges):

$$\begin{aligned} x_3 &= -\sqrt{7 - 5e^{-2.628345-1}} \approx -2.613872 \\ x_4 &= -\sqrt{7 - 5e^{-2.613872-1}} \approx -2.613306 \\ x_5 &= -\sqrt{7 - 5e^{-2.613306-1}} \approx -2.613284 \\ x_6 &= -\sqrt{7 - 5e^{-2.613284-1}} \approx -2.613283 \\ x_7 &= -\sqrt{7 - 5e^{-2.613283-1}} \approx -2.613283 \end{aligned}$$

The values for x_2 and the converged value β to 6 decimal places are:
$\boxed{x_2 = -2.628345, \beta = -2.613283}$

WMA11_P3(IAL)_Summer_2020_Q7

Solution

1. Harmonic Form Expression

To express $\cos x + 4\sin x$ in the form $R\cos(x-\alpha)$, we use the **harmonic addition theorem**. Expanding the target form using the **cosine addition formula**:

$$R\cos(x-\alpha) = R(\cos x\cos\alpha + \sin x\sin\alpha)$$

By comparing the coefficients of $\cos x$ and $\sin x$ with the original expression $\cos x + 4\sin x$:

- $R\cos\alpha = 1$
- $R\sin\alpha = 4$

To find the amplitude R:

$$\begin{aligned} R^2 &= 1^2 + 4^2 \\ R^2 &= 17 \\ R &= \sqrt{17} \end{aligned}$$

To find the phase angle α:

$$\begin{aligned} \tan\alpha &= \frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{1} \\ \alpha &= \arctan(4) \end{aligned}$$

Given $0 < \alpha < \frac{\pi}{2}$, we calculate the value in radians:

$$\alpha \approx 1.32581766...$$

Rounding to 3 decimal places, we obtain $\alpha = 1.326\,\text{rad}$.

Thus, the expression is:

$$\boxed{\sqrt{17}\cos(x-1.326)}$$

2. Minimum Height of the Seabird

The height H is modeled by:

$$H = \frac{24}{3+\cos\left(\frac{1}{2}t\right)+4\sin\left(\frac{1}{2}t\right)}$$

Using the result from part (a) where $x = \frac{1}{2}t$:

$$H = \frac{24}{3+\sqrt{17}\cos\left(\frac{1}{2}t-1.3258...\right)}$$

The height H is minimized when the denominator is maximized. The maximum value of the **cosine function** is 1.

$$H_{\min} = \frac{24}{3+\sqrt{17}(1)}$$
$$= \frac{24}{3+\sqrt{17}}$$

Calculating the numerical value:

$$H_{\min} \approx 3.3692\ldots\,\mathrm{m}$$

To convert to the nearest cm:

$$3.3692\ldots \times 100 = 336.92\ldots\,\mathrm{cm}$$

Rounding to the nearest cm gives $337\,\mathrm{cm}$.

$$\boxed{337\,\mathrm{cm}}$$

3. Value of t when $H = 10$

We set the height equation equal to 10:

$$10 = \frac{24}{3+\sqrt{17}\cos\left(\frac{1}{2}t - 1.3258\ldots\right)}$$

Rearranging to solve for the cosine term:

$$3+\sqrt{17}\cos\left(\frac{1}{2}t - 1.3258\ldots\right) = \frac{24}{10}$$
$$\sqrt{17}\cos\left(\frac{1}{2}t - 1.3258\ldots\right) = 2.4 - 3$$
$$\sqrt{17}\cos\left(\frac{1}{2}t - 1.3258\ldots\right) = -0.6$$
$$\cos\left(\frac{1}{2}t - 1.3258\ldots\right) = -\frac{0.6}{\sqrt{17}}$$

Let $\theta = \frac{1}{2}t - 1.3258\ldots$. We solve $\cos\theta = -0.14552\ldots$:

$$\theta = \arccos(-0.14552\ldots) \approx 1.7167\ldots\,\mathrm{rad}$$

Now, solve for t:

$$\frac{1}{2}t - 1.325817\ldots = 1.716757\ldots$$
$$\frac{1}{2}t = 1.716757\ldots + 1.325817\ldots$$
$$\frac{1}{2}t = 3.042574\ldots$$
$$t = 6.085148\ldots$$

Checking the constraint $0 \le t \le 6.5$, the value $t \approx 6.09$ is within the valid range. (Note: The other solution for θ would be $2\pi - 1.7167\ldots \approx 4.566$, which leads to $t \approx 11.78$, exceeding the domain).

Rounding to 2 decimal places:

$$\boxed{6.09\,\mathrm{s}}$$

WMA11_P3(IAL)_Summer_2020_Q8

Solution

1. Differentiation of $g(x)$

The function is given by $g(x) = e^{3x}\sec(2x)$ for $-\frac{\pi}{4} < x < \frac{\pi}{4}$. To find $g'(x)$, we apply the **product rule**, which states that for $u(x)v(x)$, the derivative is $u'v + uv'$.

- Let $u = e^{3x}$, then $\frac{du}{dx} = 3e^{3x}$ using the **chain rule**.
- Let $v = \sec(2x)$, then $\frac{dv}{dx} = 2\sec(2x)\tan(2x)$ using the **chain rule**.

Applying the product rule:

$$\begin{aligned} g'(x) &= (3e^{3x})(\sec(2x)) + (e^{3x})(2\sec(2x)\tan(2x)) \\ &= e^{3x}\sec(2x)[3 + 2\tan(2x)] \end{aligned}$$

$$\boxed{g'(x) = e^{3x}\sec(2x)(3 + 2\tan(2x))}$$

2. Finding the stationary point

A **stationary point** occurs where the first derivative is zero, $g'(x) = 0$.

$$e^{3x}\sec(2x)(3 + 2\tan(2x)) = 0$$

Since $e^{3x} \neq 0$ for all real x and $\sec(2x) = \frac{1}{\cos(2x)}$ is never zero within the given domain $-\frac{\pi}{4} < x < \frac{\pi}{4}$, we must have:

$$\begin{aligned} 3 + 2\tan(2x) &= 0 \\ 2\tan(2x) &= -3 \\ \tan(2x) &= -1.5 \\ 2x &= \arctan(-1.5) \\ x &= \frac{1}{2}\arctan(-1.5) \end{aligned}$$

Using a calculator for the numerical value:

$$\begin{aligned} 2x &\approx -0.98279... \\ x &\approx -0.49139... \end{aligned}$$

$$\boxed{x = \frac{1}{2}\arctan(-1.5) \approx -0.491}$$

3. Implicit differentiation and expression for dy/dx

Given the equation $x = \ln(\sin y)$ for $0 < y < \frac{\pi}{2}$, we differentiate both sides with respect to x using **implicit differentiation**:

$$\frac{d}{dx}(x) = \frac{d}{dx}(\ln(\sin y))$$
$$1 = \frac{1}{\sin y} \cdot \cos y \cdot \frac{dy}{dx}$$
$$1 = \cot y \cdot \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cot y} = \tan y$$

We need to express $\tan y$ in terms of e^x. From the original equation:

$$x = \ln(\sin y)$$
$$e^x = \sin y$$

Using the **Pythagorean identity** $\sin^2 y + \cos^2 y = 1$:

$$\cos^2 y = 1 - \sin^2 y$$
$$\cos y = \sqrt{1 - (e^x)^2}$$

(Note: $\cos y > 0$ because $0 < y < \frac{\pi}{2}$). Now, substitute these into the expression for $\tan y$:

$$\frac{dy}{dx} = \frac{\sin y}{\cos y}$$
$$= \frac{e^x}{\sqrt{1 - e^{2x}}}$$

Comparing this to the required form $\frac{dy}{dx} = \frac{e^x}{f(x)}$, we identify $f(x)$.

$\boxed{f(x) = \sqrt{1 - e^{2x}}}$

WMA11_P3(IAL)_Summer_2020_Q9

Solution

The problem involves analyzing a rational function $g(x)$, finding the equation of a tangent line, and calculating the area of a region bounded by the curve.

1. Algebraic Simplification of the Function

Given the identity for $x > -3$:

$$\frac{x^4 - x^3 - 10x^2 + 3x - 9}{x^2 - x - 12} \equiv x^2 + P + \frac{Q}{x-4}$$

We perform **polynomial long division** or use the method of equating coefficients. First, we factor the denominator:

$$x^2 - x - 12 = (x-4)(x+3)$$

Dividing the numerator $x^4 - x^3 - 10x^2 + 3x - 9$ by $x^2 - x - 12$:

- $x^2(x^2 - x - 12) = x^4 - x^3 - 12x^2$
- Subtracting this from the numerator: $(x^4 - x^3 - 10x^2 + 3x - 9) - (x^4 - x^3 - 12x^2) = 2x^2 + 3x - 9$
- $2(x^2 - x - 12) = 2x^2 - 2x - 24$
- Subtracting this: $(2x^2 + 3x - 9) - (2x^2 - 2x - 24) = 5x + 15$

Thus, the expression becomes:

$$x^2 + 2 + \frac{5x + 15}{(x-4)(x+3)}$$

Simplifying the remainder term:

$$\frac{5(x+3)}{(x-4)(x+3)} = \frac{5}{x-4}$$

Comparing this to the form $x^2 + P + \frac{Q}{x-4}$, we find:

$$\boxed{P = 2, \quad Q = 5}$$

2. Equation of the Tangent at $x = 2$

The curve C is defined by $g(x) = x^2 + 2 + \frac{5}{x-4}$ for $-3 < x < 3.5$.

- **Point of tangency:** At $x = 2$:

$$g(2) = 2^2 + 2 + \frac{5}{2-4} = 4 + 2 - \frac{5}{2} = 6 - 2.5 = 3.5$$

  The point is $(2, 3.5)$.

- **Gradient of the tangent:** We find the **derivative** $g'(x)$:

$$g'(x) = \frac{d}{dx}(x^2 + 2 + 5(x-4)^{-1}) = 2x - \frac{5}{(x-4)^2}$$

At $x = 2$:

$$m = g'(2) = 2(2) - \frac{5}{(2-4)^2} = 4 - \frac{5}{4} = \frac{16-5}{4} = 2.75$$

- **Equation of the line:** Using the point-slope form $y - y_1 = m(x - x_1)$:

$$y - 3.5 = 2.75(x-2)$$

$$y = 2.75x - 5.5 + 3.5$$

$$y = 2.75x - 2$$

In the form $y = mx + c$:

$$\boxed{y = \frac{11}{4}x - 2}$$

3. Exact Area of Region R

The region R is bounded by C, the y-axis ($x = 0$), the x-axis, and $x = 2$. The area A is given by the **definite integral**:

$$A = \int_0^2 g(x)\,dx = \int_0^2 \left(x^2 + 2 + \frac{5}{x-4}\right)dx$$

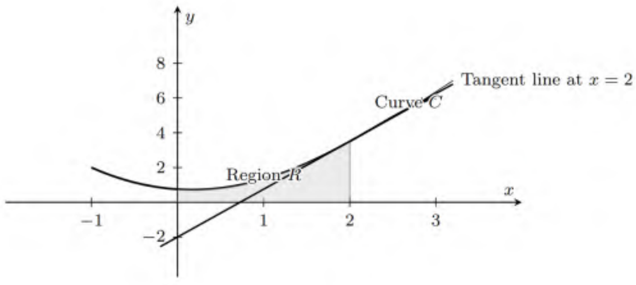
Integrating term by term:

$$A = \left[\frac{1}{3}x^3 + 2x + 5\ln|\ x - 4\ |\right]_0^2$$
$$= \left(\frac{1}{3}(2)^3 + 2(2) + 5\ln|\ 2-4\ |\right) - \left(\frac{1}{3}(0)^3 + 2(0) + 5\ln|\ 0-4\ |\right)$$
$$= \left(\frac{8}{3} + 4 + 5\ln 2\right) - (5\ln 4)$$

Using the **logarithm power rule** $\ln 4 = \ln(2^2) = 2\ln 2$:

$$A = \frac{8+12}{3} + 5\ln 2 - 10\ln 2$$
$$= \frac{20}{3} - 5\ln 2$$

The constants are $a = \frac{20}{3}$ and $b = -5$.



$$\boxed{\frac{20}{3} - 5\ln 2}$$

WMA11_P3(IAL)_Winter_2020_Q1

Solution

The population of toads N at time t (in years) is given by the **logistic growth model**:

$$N = \frac{900e^{0.12t}}{2e^{0.12t} + 1}, \quad t \geq 0$$

1. Initial population calculation - The start of the study corresponds to $t = 0$. Substituting this value into the equation:

$$\begin{aligned} N(0) &= \frac{900e^{0.12(0)}}{2e^{0.12(0)} + 1} \\ &= \frac{900(1)}{2(1) + 1} \\ &= \frac{900}{3} \\ &= 300 \end{aligned}$$

- The number of toads at the start of the study is $\boxed{300}$.

2. Time required to reach 420 toads - We set $N = 420$ and solve for t:

$$\begin{aligned} 420 &= \frac{900e^{0.12t}}{2e^{0.12t} + 1} \\ 420(2e^{0.12t} + 1) &= 900e^{0.12t} \\ 840e^{0.12t} + 420 &= 900e^{0.12t} \\ 420 &= 900e^{0.12t} - 840e^{0.12t} \\ 420 &= 60e^{0.12t} \\ e^{0.12t} &= \frac{420}{60} \\ e^{0.12t} &= 7 \end{aligned}$$

- Taking the **natural logarithm** ($\ln$) of both sides:

$$\begin{aligned} 0.12t &= \ln(7) \\ t &= \frac{\ln(7)}{0.12} \\ t &\approx 16.21548\ldots \end{aligned}$$

- Rounding to 2 decimal places, we find $t \approx 16.22$. - The value of t is $\boxed{16.22}$.

3. Analysis of the population limit - To determine the maximum possible population, we examine the **horizontal asymptote** by taking the limit as $t \to \infty$. - Divide the numerator and denominator by $e^{0.12t}$:

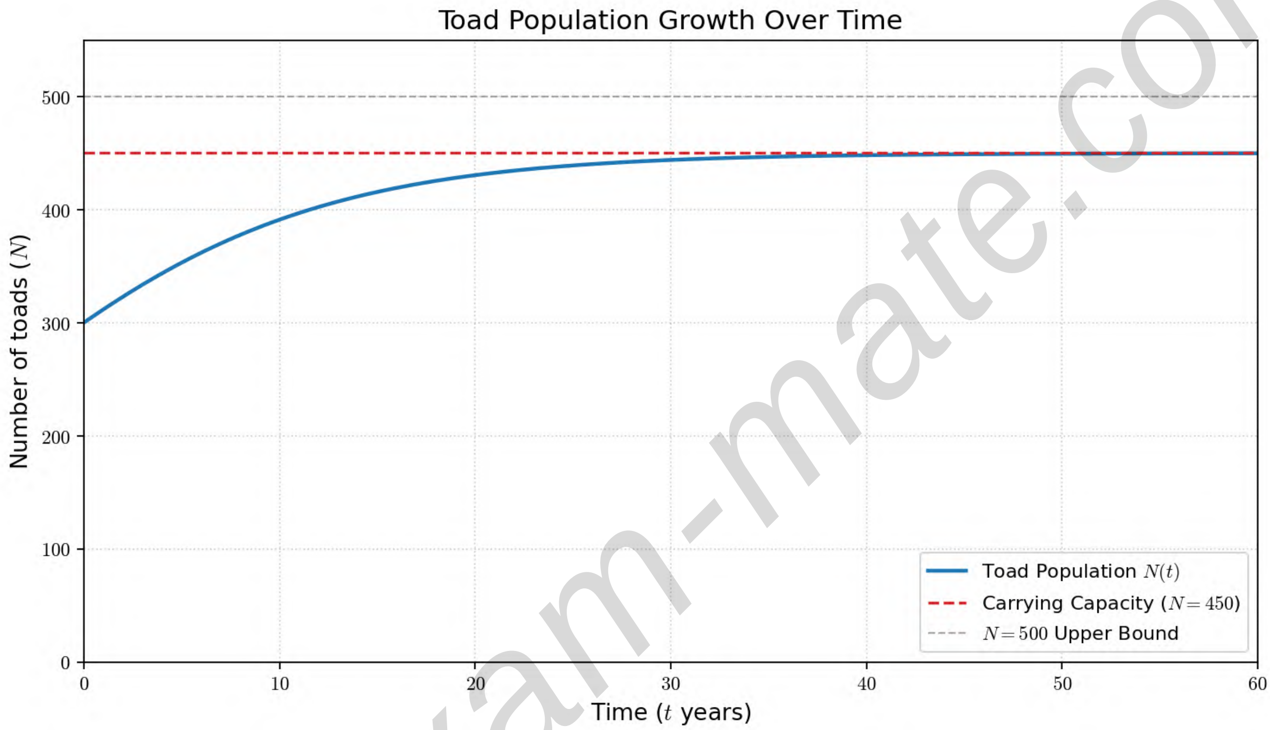
$$N = \frac{900}{2 + \frac{1}{e^{0.12t}}}$$

$$\lim_{t \to \infty} N = \lim_{t \to \infty} \frac{900}{2 + e^{-0.12t}}$$

$$= \frac{900}{2 + 0}$$

$$= 450$$

- Since the **carrying capacity** of the model is 450, the population N approaches 450 from below as t increases. Because $450 < 500$, the number of toads can never reach 500.



WMA11_P3(IAL)_Winter_2020_Q2

Solution

The functions f and g are defined as:

$$f(x) = \frac{12}{x+1}, \quad x > 0, x \in \mathbb{R}$$
$$g(x) = \frac{5}{2}\ln x, \quad x > 0, x \in \mathbb{R}$$

1. Evaluation of the composite function $fg(e^2)$

- First, evaluate the inner function $g(x)$ at $x = e^2$:

$$\begin{aligned} g(e^2) &= \frac{5}{2}\ln(e^2) \\ &= \frac{5}{2}\cdot 2 \\ &= 5 \end{aligned}$$

- Next, substitute this result into the outer function $f(x)$:

$$\begin{aligned} f(g(e^2)) &= f(5) \\ &= \frac{12}{5+1} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

The value of $fg(e^2)$ is $\boxed{2}$.

2. Finding the inverse function f^{-1}

- To find the **inverse function**, let $y = f(x)$ and solve for x in terms of y:

$$\begin{aligned} y &= \frac{12}{x+1} \\ y(x+1) &= 12 \\ x+1 &= \frac{12}{y} \\ x &= \frac{12}{y} - 1 \end{aligned}$$

- Interchanging x and y to express the inverse in terms of x:

$$f^{-1}(x) = \frac{12}{x} - 1$$

- To determine the **domain** of f^{-1}, we find the **range** of f. Since $x > 0$, the denominator $x+1$ is always greater than 1. As $x \to 0^+$, $f(x) \to 12$. As $x \to \infty$, $f(x) \to 0$. Thus, the range of f is $0 < y < 12$. The inverse function is $\boxed{f^{-1}(x) = \frac{12}{x} - 1, \quad 0 < x < 12}$.

3. Solving the equation $f^{-1}(x) = f(x)$

- Equating the two expressions:

$$\frac{12}{x} - 1 = \frac{12}{x+1}$$

- Multiply the entire equation by $x(x+1)$ to clear the denominators:

$$12(x+1) - x(x+1) = 12x$$
$$12x + 12 - x^2 - x = 12x$$
$$12 - x^2 - x = 0$$
$$x^2 + x - 12 = 0$$

- Factor the **quadratic equation**:

$$(x+4)(x-3) = 0$$

- This yields two potential solutions: $x = -4$ or $x = 3$.
- We must check the constraints. The domain of f is $x > 0$, and the domain of f^{-1} (which is the range of f) is $0 < x < 12$.
- $x = -4$ is outside the valid domain.
- $x = 3$ satisfies $0 < 3 < 12$.

The only real solution is $\boxed{x = 3}$.

WMA11_P3(IAL)_Winter_2020_Q3

Solution

1. Equation linking $\log_{10} y$ and $\log_{10} x$

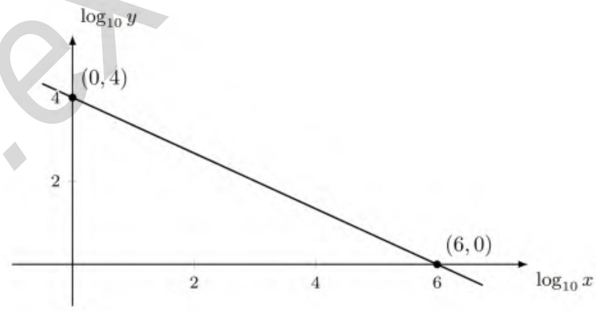
The graph shows a linear relationship between the variables $Y = \log_{10} y$ and $X = \log_{10} x$. The line passes through the points $(0, 4)$ and $(6, 0)$.

- **Determine the gradient (m):** Using the **gradient formula** $m = \frac{Y_2 - Y_1}{X_2 - X_1}$:

$$m = \frac{0-4}{6-0} = -\frac{4}{6} = -\frac{2}{3}$$

- **Determine the vertical intercept (c):** The line crosses the vertical axis at $(0, 4)$, so $c = 4$.
- **Formulate the linear equation:** Using the **slope-intercept form** $Y = mX + c$:

$$\log_{10} y = -\frac{2}{3} \log_{10} x + 4$$



2. Expressing y in the form px^q

To find the relationship between y and x, we apply **logarithmic identities** to the equation derived in part (a).

- **Apply the power law of logarithms:**

$$\log_{10} y = \log_{10}(x^{-2/3}) + 4$$

- **Express the constant as a logarithm:** Since $4 = \log_{10}(10^4) = \log_{10}(10000)$:

$$\log_{10} y = \log_{10}(x^{-2/3}) + \log_{10}(10000)$$

- **Apply the product law of logarithms:**

$$\log_{10} y = \log_{10}(10000 \cdot x^{-2/3})$$

- **Remove logarithms by taking the antilogarithm of both sides:**

$$y = 10000x^{-2/3}$$

Comparing this to the form $y = px^q$, we identify the constants:

- $p = 10000$
- $q = -\frac{2}{3}$

Final Answer: (a) $\boxed{\log_{10} y = -\frac{2}{3}\log_{10} x + 4}$ (b) $\boxed{y = 10000x^{-2/3}}$

WMA11_P3(IAL)_Winter_2020_Q4

Solution

1. Differentiation of $f(x)$

To find the derivative of $f(x) = \frac{(2x+5)^2}{x-3}$, we apply the **quotient rule**:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let $u = (2x+5)^2$ and $v = x - 3$.

- Using the **chain rule**, $\frac{du}{dx} = 2(2x+5) \cdot 2 = 4(2x+5)$.
- $\frac{dv}{dx} = 1$.

Substituting these into the quotient rule formula:

$$\begin{aligned} f'(x) &= \frac{(x-3) \cdot 4(2x+5) - (2x+5)^2 \cdot 1}{(x-3)^2} \\ &= \frac{(2x+5)[4(x-3) - (2x+5)]}{(x-3)^2} \\ &= \frac{(2x+5)[4x - 12 - 2x - 5]}{(x-3)^2} \\ &= \frac{(2x+5)(2x-17)}{(x-3)^2} \end{aligned}$$

The derivative is in the form $\frac{P(x)}{Q(x)}$ where $P(x) = (2x+5)(2x-17)$ and $Q(x) = (x-3)^2$.

2. Range of values for which $f(x)$ is increasing

A function is **increasing** when its first derivative is greater than or equal to zero ($f'(x) \geq 0$). Since the denominator $(x-3)^2$ is always positive for $x \neq 3$, the sign of $f'(x)$ depends solely on the numerator $P(x) = (2x+5)(2x-17)$.

- We solve the inequality $(2x+5)(2x-17) \geq 0$.
- The roots of the quadratic are $x = -2.5$ and $x = 8.5$.
- For a quadratic with a positive leading coefficient, the expression is non-negative outside the roots.

Thus, the range of values is:

$$\boxed{x \leq -2.5 \quad \text{or} \quad x \geq 8.5}$$

3. Maximum of $g(x)$

Given $g(x) = x\sqrt{\sin 4x}$ for $0 \leq x < \frac{\pi}{4}$. To find the maximum, we set $g'(x) = 0$ using the **product rule** and the chain rule:

$$g'(x) = 1 \cdot \sqrt{\sin 4x} + x \cdot \frac{1}{2\sqrt{\sin 4x}} \cdot (\cos 4x \cdot 4)$$
$$= \sqrt{\sin 4x} + \frac{2x \cos 4x}{\sqrt{\sin 4x}}$$

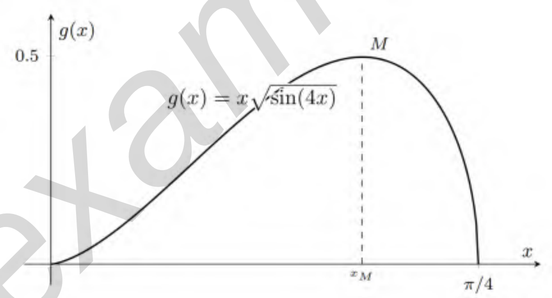
Setting $g'(x) = 0$:

$$\sqrt{\sin 4x} + \frac{2x \cos 4x}{\sqrt{\sin 4x}} = 0$$
$$\frac{\sin 4x + 2x \cos 4x}{\sqrt{\sin 4x}} = 0$$
$$\sin 4x + 2x \cos 4x = 0$$

To transform this into the form $\tan 4x + kx = 0$, we divide the entire equation by $\cos 4x$ (noting that $\cos 4x \neq 0$ in the given interval for a maximum):

$$\frac{\sin 4x}{\cos 4x} + \frac{2x \cos 4x}{\cos 4x} = 0$$
$$\tan 4x + 2x = 0$$

Comparing this to the required form $\tan 4x + kx = 0$, we find the constant k.



$$\boxed{k = 2}$$

WMA11_P3(IAL)_Winter_2020_Q5

Solution

1. Transformation to a Polynomial Equation

To transform the given trigonometric equation into a polynomial in terms of $t = \tan x$, we apply standard **trigonometric identities**.

- First, we express $\tan 2x$ using the **double-angle formula**:

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} = \frac{2t}{1 - t^2}$$

- Next, we express $\cot x$ and $\sec^2 x$ in terms of t:
  - $\cot x = \frac{1}{\tan x} = \frac{1}{t}$
  - $\sec^2 x = 1 + \tan^2 x = 1 + t^2$

Substituting these into the original equation $12\tan 2x + 5\cot x \sec^2 x = 0$:

$$12\left(\frac{2t}{1 - t^2}\right) + 5\left(\frac{1}{t}\right)(1 + t^2) = 0$$

$$\frac{24t}{1 - t^2} + \frac{5(1 + t^2)}{t} = 0$$

To eliminate the denominators, we multiply the entire equation by $t(1 - t^2)$:

$$24t(t) + 5(1 + t^2)(1 - t^2) = 0$$

$$24t^2 + 5(1 - t^4) = 0$$

$$24t^2 + 5 - 5t^4 = 0$$

Rearranging the terms to match the required form:

$$5t^4 - 24t^2 - 5 = 0$$

2. Solving the Polynomial Equation

The equation $5t^4 - 24t^2 - 5 = 0$ is a **quadratic equation** in terms of $u = t^2$.

$$5u^2 - 24u - 5 = 0$$

Using the **quadratic formula** $u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$u = \frac{24 \pm \sqrt{(-24)^2 - 4(5)(-5)}}{2(5)}$$

$$= \frac{24 \pm \sqrt{576 + 100}}{10}$$

$$= \frac{24 \pm \sqrt{676}}{10}$$

$$= \frac{24 \pm 26}{10}$$

This gives two possible values for u:

- $u_1 = \frac{24+26}{10} = \frac{50}{10} = 5$
- $u_2 = \frac{24-26}{10} = \frac{-2}{10} = -0.2$

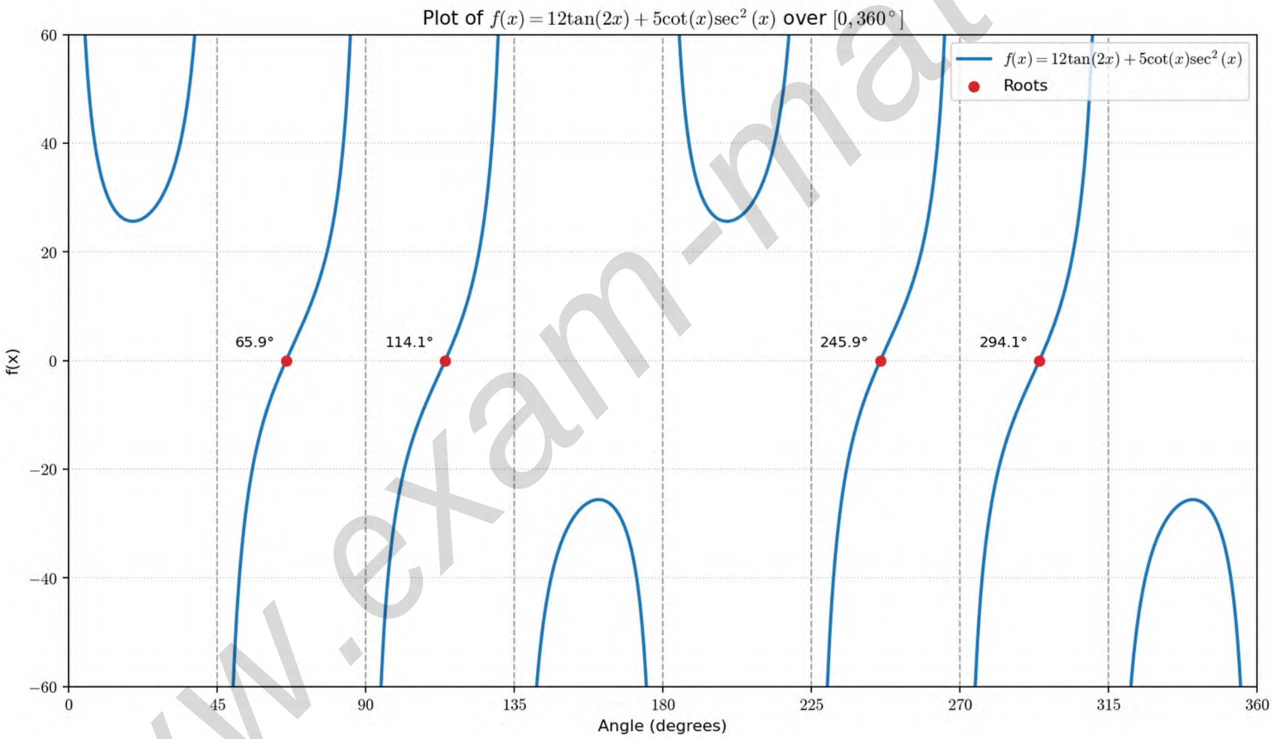
Since $u = t^2 = \tan^2 x$, and the square of a real number cannot be negative, we discard $u = -0.2$. Thus:

$$t^2 = 5 \implies t = \pm\sqrt{5}$$

3. Finding the Values of x

We solve for x in the interval $0^\circ \le x < 360^\circ$ for both cases of $t = \tan x$.

- **Case 1: $\tan x = \sqrt{5}$**
  - Principal value: $x = \arctan(\sqrt{5}) \approx 65.9^\circ$
  - Second solution in the interval: $x = 180^\circ + 65.9^\circ = 245.9^\circ$
- **Case 2: $\tan x = -\sqrt{5}$**
  - Principal value: $x = \arctan(-\sqrt{5}) \approx -65.9^\circ$
  - First solution in the interval: $x = 180^\circ - 65.9^\circ = 114.1^\circ$
  - Second solution in the interval: $x = 360^\circ - 65.9^\circ = 294.1^\circ$



The solutions for x to one decimal place are:

$$\boxed{x = 65.9^\circ, 114.1^\circ, 245.9^\circ, 294.1^\circ}$$

WMA11_P3(IAL)_Winter_2020_Q6

Solution

The function is defined as $f(x) = 2 \mid 2x - 5 \mid + 3$ for $x \geq 0$.

1. Coordinates of the vertex P The **vertex** of an absolute value function of the form $y = a \mid bx - c \mid + d$ occurs where the argument of the absolute value is zero.

- Setting the argument to zero:

$$\begin{aligned} 2x - 5 &= 0 \\ x &= 2.5 \end{aligned}$$

- Substituting $x = 2.5$ into $f(x)$:

$$\begin{aligned} f(2.5) &= 2 \mid 2(2.5) - 5 \mid + 3 \\ &= 2 \mid 0 \mid + 3 \\ &= 3 \end{aligned}$$

  The coordinates of P are $(2.5, 3)$.

$\boxed{P(2.5, 3)}$

2. Solving the equation $f(x) = 3x - 2$ We must solve $2 \mid 2x - 5 \mid + 3 = 3x - 2$, which simplifies to $2 \mid 2x - 5 \mid = 3x - 5$.

- **Case 1:** $2x - 5 \geq 0$ (i.e., $x \geq 2.5$)

$$\begin{aligned} 2(2x - 5) &= 3x - 5 \\ 4x - 10 &= 3x - 5 \\ x &= 5 \end{aligned}$$

  Since $5 \geq 2.5$, this is a valid solution.

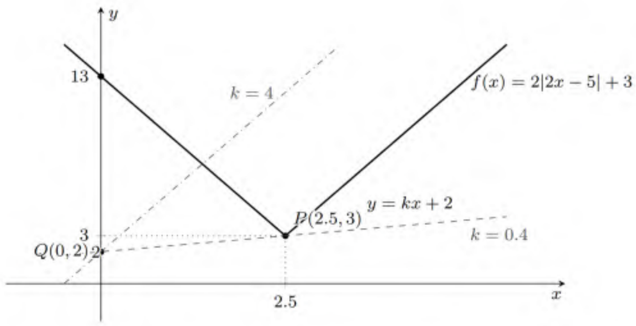
- **Case 2:** $2x - 5 < 0$ (i.e., $x < 2.5$)

$$\begin{aligned} -2(2x - 5) &= 3x - 5 \\ -4x + 10 &= 3x - 5 \\ 7x &= 15 \\ x &= \frac{15}{7} \approx 2.14 \end{aligned}$$

  Since $2.14 < 2.5$, this is also a valid solution.

$\boxed{x = \frac{15}{7}, x = 5}$

3. Range of values for k such that $f(x) = kx + 2$ has exactly two roots The equation $f(x) = kx + 2$ represents the intersection of the V-shaped graph $y = f(x)$ and a line $y = kx + 2$ passing through the fixed **y-intercept** $(0, 2)$.



- **Lower Bound:** The line must have a slope greater than the slope of the line connecting $(0, 2)$ and the vertex $P(2.5, 3)$.

$$k_1 = \frac{3-2}{2.5-0}$$
$$= \frac{1}{2.5} = 0.4$$

  If $k = 0.4$, there is exactly one root (at the vertex). For $k > 0.4$, the line intersects both branches of the absolute value function.

- **Upper Bound:** The line must not be parallel to or steeper than the right branch of $f(x)$. The right branch for $x > 2.5$ is $y = 2(2x-5)+3 = 4x-7$. The slope of this branch is 4. If $k \geq 4$, the line will only intersect the left branch once (or not at all if it's too steep), as it will never "catch up" to the right branch. Thus, we require $k < 4$.

- **Left Branch Check:** The left branch for $0 \leq x < 2.5$ is $y = -4x + 13$. Since the line $y = kx + 2$ starts at $y = 2$ (below the branch's start at $y = 13$) and k is positive, it will always intersect the left branch once as long as $k > -4$.

Therefore, for exactly two intersections, the slope k must be strictly between the slope to the vertex and the slope of the right branch.

$\boxed{0.4 < k < 4}$

WMA11_P3(IAL)_Winter_2020_Q7

Solution

The curve is defined by the equation:

$$f(x) = 2\cos 3x - 3x + 4, \quad x > 0$$

where x is measured in radians.

1. Verification of the root α To show that the root α lies between 0.8 and 0.9, we apply the **Intermediate Value Theorem** by evaluating the function at the boundaries.

- At $x = 0.8$:

$$\begin{aligned} f(0.8) &= 2\cos(3 \cdot 0.8) - 3(0.8) + 4 \\ &= 2\cos(2.4) - 2.4 + 4 \\ &\approx 2(-0.73739) + 1.6 \\ &\approx 0.1252 \end{aligned}$$

- At $x = 0.9$:

$$\begin{aligned} f(0.9) &= 2\cos(3 \cdot 0.9) - 3(0.9) + 4 \\ &= 2\cos(2.7) - 2.7 + 4 \\ &\approx 2(-0.90407) + 1.3 \\ &\approx -0.5081 \end{aligned}$$

  Since $f(0.8) > 0$ and $f(0.9) < 0$, and the function is continuous, there must be at least one root α such that $0.8 < \alpha < 0.9$.

2. Iterative Approximation The given **iteration formula** is:

$$x_{n+1} = \frac{1}{3}\arccos(1.5x_n - 2)$$

Starting with $x_1 = 0.8$:

- (i) Finding x_2:

$$\begin{aligned} x_2 &= \frac{1}{3}\arccos(1.5(0.8) - 2) \\ &= \frac{1}{3}\arccos(1.2 - 2) \\ &= \frac{1}{3}\arccos(-0.8) \\ &\approx 0.81041 \end{aligned}$$

  Rounding to 4 decimal places, we get $x_2 = 0.8104$.
- (ii) Finding x_5: Continuing the iterations:

$$x_3 = \frac{1}{3}\arccos(1.5(0.81041) - 2) \approx 0.8069$$
$$x_4 = \frac{1}{3}\arccos(1.5(0.8069) - 2) \approx 0.8081$$
$$x_5 = \frac{1}{3}\arccos(1.5(0.8081) - 2) \approx 0.8077$$

Rounding to 4 decimal places, we get $x_5 = 0.8077$.

3. Finding Local Minima using Calculus To find the **stationary points**, we differentiate $f(x)$ with respect to x:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2\cos 3x - 3x + 4)\\ &= -6\sin 3x - 3\end{aligned}$$

Setting the derivative to zero for **local minima**:

$$\begin{aligned}-6\sin 3x - 3 &= 0\\ \sin 3x &= -0.5\end{aligned}$$

The general solution for $3x$ where $\sin\theta = -0.5$ is:

$$3x = \frac{7\pi}{6} + 2k\pi \quad \text{or} \quad 3x = \frac{11\pi}{6} + 2k\pi$$

To identify which are minima, we check the **second derivative**:

$$\frac{d^2y}{dx^2} = -18\cos 3x$$

For a local minimum, we require $\frac{d^2y}{dx^2} > 0$, which implies $\cos 3x < 0$.

- For $3x = \frac{7\pi}{6}$, $\cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} < 0$. This is a minimum.
- For $3x = \frac{11\pi}{6}$, $\cos(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2} > 0$. This is a maximum.

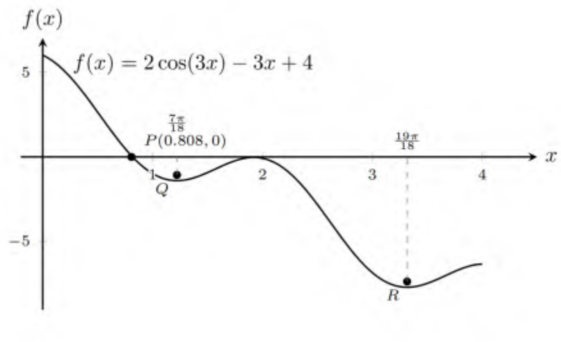
The two smallest values of x for local minima occur when $3x = \frac{7\pi}{6}$ and $3x = \frac{7\pi}{6} + 2\pi = \frac{19\pi}{6}$.

- For point Q ($x = \beta$):

$$\beta = \frac{7\pi}{18}$$

- For point R ($x = \lambda$):

$$\lambda = \frac{19\pi}{18}$$



(a) $f(0.8) \approx 0.1252$, $f(0.9) \approx -0.5081$. Change of sign implies root in interval.

(b) (i) $\boxed{x_2 = 0.8104}$ (ii) $\boxed{x_5 = 0.8077}$

(c) $\boxed{\beta = \frac{7\pi}{18}, \lambda = \frac{19\pi}{18}}$

WMA11_P3(IAL)_Winter_2020_Q8

Solution

1. Evaluation of the Definite Integral

To find the exact value of the integral $I = \int_3^{42} \frac{2}{3x-1}\,dx$, we apply the **fundamental theorem of calculus** and the standard integral form for reciprocal linear functions:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|\ ax+b\ | + C$$

- **Integration:**

$$\begin{aligned}\int_3^{42} \frac{2}{3x-1}\,dx &= 2\left[\frac{1}{3}\ln|\ 3x-1\ |\right]_3^{42} \\ &= \frac{2}{3}[\ln(3(42)-1) - \ln(3(3)-1)] \\ &= \frac{2}{3}[\ln(126-1) - \ln(9-1)] \\ &= \frac{2}{3}(\ln 125 - \ln 8)\end{aligned}$$

- **Simplification:** Using the **logarithm laws**, specifically $\ln a - \ln b = \ln(\frac{a}{b})$ and $\ln(a^n) = n\ln a$:

$$\begin{aligned}I &= \frac{2}{3}\ln\left(\frac{125}{8}\right) \\ &= \frac{2}{3}\ln\left(\frac{5^3}{2^3}\right) \\ &= \frac{2}{3}\ln\left(\frac{5}{2}\right)^3 \\ &= \frac{2}{3}\cdot 3\ln\left(\frac{5}{2}\right) \\ &= 2\ln\left(\frac{5}{2}\right)\end{aligned}$$

  Alternatively, this can be expressed as $\ln(\frac{25}{4})$ or $\ln 6.25$.

$$\boxed{2\ln\left(\frac{5}{2}\right)}$$

2. Integration of the Rational Function

Given $h(x) = \frac{2x^3-7x^2+8x+1}{(x-1)^2}$ for $x > 1$, we first express it in the form $Ax + B + \frac{C}{(x-1)^2}$.

- **Partial Fraction Decomposition / Algebraic Division:** Expand the denominator: $(x-1)^2 = x^2 - 2x + 1$. Perform **polynomial long division** of $2x^3 - 7x^2 + 8x + 1$ by $x^2 - 2x + 1$:

  1. Divide $2x^3$ by x^2 to get $2x$.
  2. Multiply $2x(x^2 - 2x + 1) = 2x^3 - 4x^2 + 2x$.

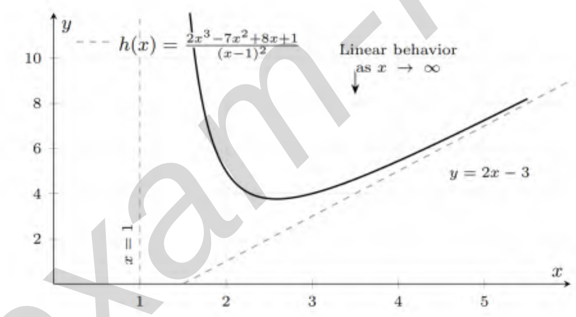
3. Subtract: $(2x^3 - 7x^2 + 8x + 1) - (2x^3 - 4x^2 + 2x) = -3x^2 + 6x + 1$.
4. Divide $-3x^2$ by x^2 to get -3.
5. Multiply $-3(x^2 - 2x + 1) = -3x^2 + 6x - 3$.
6. Subtract: $(-3x^2 + 6x + 1) - (-3x^2 + 6x - 3) = 4$.

Thus, $A = 2$, $B = -3$, and $C = 4$:

$$h(x) = 2x - 3 + \frac{4}{(x-1)^2}$$

- **Indefinite Integration:** Now, integrate $h(x)$ term by term:

$$\begin{aligned}\int h(x)\,dx &= \int (2x - 3 + 4(x-1)^{-2})\,dx \\ &= 2\left(\frac{x^2}{2}\right) - 3x + 4\left(\frac{(x-1)^{-1}}{-1}\right) + C \\ &= x^2 - 3x - \frac{4}{x-1} + C\end{aligned}$$



$$\boxed{x^2 - 3x - \frac{4}{x-1} + C}$$

WMA11_P3(IAL)_Winter_2020_Q9

Solution

1. Harmonic Addition Theorem

To express $f(\theta) = 5\cos\theta - 4\sin\theta$ in the form $R\cos(\theta + \alpha)$, we use the **harmonic addition theorem** (or R-formula). We expand the target expression using the **cosine addition formula**:

$$R\cos(\theta + \alpha) = R(\cos\theta\cos\alpha - \sin\theta\sin\alpha) = (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$$

By comparing the coefficients of $\cos\theta$ and $\sin\theta$ with $f(\theta) = 5\cos\theta - 4\sin\theta$:

- $R\cos\alpha = 5$
- $R\sin\alpha = 4$

To find the exact value of R:

$$\begin{aligned} R^2\cos^2\alpha + R^2\sin^2\alpha &= 5^2 + 4^2 \\ R^2(\cos^2\alpha + \sin^2\alpha) &= 25 + 16 \\ R^2 &= 41 \\ R &= \sqrt{41} \end{aligned}$$

To find the value of α:

$$\begin{aligned} \frac{R\sin\alpha}{R\cos\alpha} &= \frac{4}{5} \\ \tan\alpha &= 0.8 \\ \alpha &= \arctan(0.8) \end{aligned}$$

Using a calculator in radian mode:

$$\alpha \approx 0.6747409... \approx 0.675\,\mathrm{rad}$$

Thus, $f(\theta) = \sqrt{41}\cos(\theta + 0.675)$.

$\boxed{R = \sqrt{41}, \alpha = 0.675}$

2. Geometric Transformations

The curve $y = \cos\theta$ is transformed into $y = \sqrt{41}\cos(\theta + 0.675)$ via a sequence of two transformations.

- **(i) Stretch:** The coefficient $R = \sqrt{41}$ multiplying the function represents a **vertical stretch**. The transformation is a stretch parallel to the y-axis (or in the y-direction) with a **scale factor** of $\sqrt{41}$.
- **(ii) Translation:** The term $(\theta + \alpha)$ inside the argument represents a **horizontal translation**. The transformation is a translation by the vector $\binom{-0.675}{0}$, which corresponds to a shift of 0.675 units in the negative θ-direction (to the left).

3. Range of the Function g

Given $g(\theta) = \frac{90}{4+(f(\theta))^2}$, we determine the range by analyzing the bounds of $f(\theta)$. From part (a), $f(\theta) = \sqrt{41}\cos(\theta + \alpha)$. Since the range of the cosine function is $[-1, 1]$, the range of $f(\theta)$ is $[-\sqrt{41}, \sqrt{41}]$.

Consequently, the term $(f(\theta))^2$ varies as follows:

- Minimum value: $(f(\theta))^2 = 0$ (when $\cos(\theta + \alpha) = 0$)
- Maximum value: $(f(\theta))^2 = (\pm\sqrt{41})^2 = 41$ (when $\cos(\theta + \alpha) = \pm 1$)

Now we calculate the stationary values of $g(\theta)$:

- When $(f(\theta))^2 = 0$:

$$g_{\max} = \frac{90}{4+0} = \frac{90}{4} = 22.5$$

- When $(f(\theta))^2 = 41$:

$$g_{\min} = \frac{90}{4+41} = \frac{90}{45} = 2$$

Since $g(\theta)$ is a continuous function, it takes all values between these bounds.

$\boxed{2 \leq g(\theta) \leq 22.5}$

WMA11_P3(IAL)_Summer_2021_Q1

Solution

1. Derivation of the stationary point equation

To find the **stationary point** P of the curve C defined by $y = x^2 \cos(\frac{1}{2}x)$ for $0 < x \leq \pi$, we must first determine the derivative dy/dx and set it to zero.

- Applying the **product rule**, where $u = x^2$ and $v = \cos(\frac{1}{2}x)$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2) \cdot \cos\left(\frac{1}{2}x\right) + x^2 \cdot \frac{d}{dx}\left[\cos\left(\frac{1}{2}x\right)\right] \\ &= 2x\cos\left(\frac{1}{2}x\right) + x^2\left[-\frac{1}{2}\sin\left(\frac{1}{2}x\right)\right] \\ &= 2x\cos\left(\frac{1}{2}x\right) - \frac{1}{2}x^2\sin\left(\frac{1}{2}x\right)\end{aligned}$$

- At the stationary point P, $\frac{dy}{dx} = 0$:

$$2x\cos\left(\frac{1}{2}x\right) - \frac{1}{2}x^2\sin\left(\frac{1}{2}x\right) = 0$$
$$2x\cos\left(\frac{1}{2}x\right) = \frac{1}{2}x^2\sin\left(\frac{1}{2}x\right)$$

- Since $0 < x \leq \pi$, we can divide both sides by $x\cos(\frac{1}{2}x)$ (noting that $\cos(\frac{1}{2}x) \neq 0$ for $x < \pi$):

$$2 = \frac{1}{2}x\tan\left(\frac{1}{2}x\right)$$
$$\frac{4}{x} = \tan\left(\frac{1}{2}x\right)$$

- Taking the **arctangent** of both sides:

$$\frac{1}{2}x = \arctan\left(\frac{4}{x}\right)$$
$$x = 2\arctan\left(\frac{4}{x}\right)$$

2. Iterative calculation

We use the **fixed-point iteration** formula $x_{n+1} = 2\arctan\left(\frac{4}{x_n}\right)$ with the initial value $x_1 = 2$.

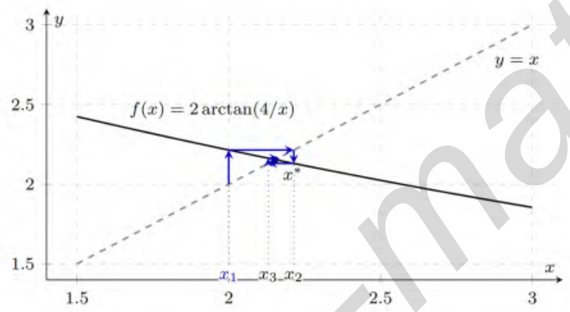
- For $n = 1$:

$$\begin{aligned}x_2 &= 2\arctan\left(\frac{4}{x_1}\right) \\ &= 2\arctan\left(\frac{4}{2}\right) \\ &= 2\arctan(2) \\ &\approx 2.214297...\end{aligned}$$

- Continuing the iteration to find x_6:
  - $x_3 = 2\arctan\left(\frac{4}{2.214297}\right) \approx 2.131614...$
  - $x_4 = 2\arctan\left(\frac{4}{2.131614}\right) \approx 2.162125...$
  - $x_5 = 2\arctan\left(\frac{4}{2.162125}\right) \approx 2.150756...$
  - $x_6 = 2\arctan\left(\frac{4}{2.150756}\right) \approx 2.154972...$

Rounding the results to 3 decimal places:

- $x_2 \approx 2.214$
- $x_6 \approx 2.155$



$$x_2 = 2.214$$
$$x_6 = 2.155$$

WMA11_P3(IAL)_Summer_2021_Q2

Solution

1. Derivation of the Trigonometric Identity

To show that $\frac{1-\cos 2x}{2\sin 2x} \equiv k\tan x$, we apply the **double-angle formulas** for sine and cosine.

- Recall the identities:
  - $\cos 2x = 1 - 2\sin^2 x$
  - $\sin 2x = 2\sin x\cos x$
- Substitute these into the left-hand side (LHS) of the expression:

$$\begin{aligned}\text{LHS} &= \frac{1-(1-2\sin^2 x)}{2(2\sin x\cos x)} \\ &= \frac{2\sin^2 x}{4\sin x\cos x} \\ &= \frac{\sin x}{2\cos x} \\ &= \frac{1}{2}\tan x\end{aligned}$$

Comparing this to the form $k\tan x$, we find the constant to be:

$$\boxed{k = \frac{1}{2}}$$

2. Solving the Trigonometric Equation

We are asked to solve the equation for $0^\circ < \theta < 90^\circ$:

$$\frac{9(1-\cos 2\theta)}{2\sin 2\theta} = 2\sec^2\theta$$

- Using the identity derived in part (a), where $\frac{1-\cos 2\theta}{2\sin 2\theta} = \frac{1}{2}\tan\theta$, the equation becomes:

$$9\left(\frac{1}{2}\tan\theta\right) = 2\sec^2\theta$$

$$\frac{9}{2}\tan\theta = 2\sec^2\theta$$

- Use the **Pythagorean identity** $\sec^2\theta = 1 + \tan^2\theta$ to express the equation in terms of $\tan\theta$:

$$\frac{9}{2}\tan\theta = 2(1+\tan^2\theta)$$

$$9\tan\theta = 4 + 4\tan^2\theta$$

$$4\tan^2\theta - 9\tan\theta + 4 = 0$$

- This is a quadratic equation in terms of $\tan\theta$. We apply the **quadratic formula**:

$$\tan\theta = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(4)(4)}}{2(4)}$$
$$= \frac{9 \pm \sqrt{81 - 64}}{8}$$
$$= \frac{9 \pm \sqrt{17}}{8}$$

- Calculate the two possible values for $\tan\theta$:
  - Case 1: $\tan\theta = \frac{9+\sqrt{17}}{8} \approx 1.640388$
  - Case 2: $\tan\theta = \frac{9-\sqrt{17}}{8} \approx 0.609612$
- Solve for θ within the interval $0^\circ < \theta < 90^\circ$:
  - $\theta_1 = \arctan(1.640388) \approx 58.634^\circ$
  - $\theta_2 = \arctan(0.609612) \approx 31.366^\circ$

Rounding to one decimal place as requested:

$$\boxed{\theta = 31.4^\circ, 58.6^\circ}$$

WMA11_P3(IAL)_Summer_2021_Q3

Solution

1. Indefinite Integration of a Rational Function

To find the integral $\int \frac{12}{(2x-1)^2} dx$, we apply the **power rule for integration** combined with the **reverse chain rule**.

- First, rewrite the integrand using a negative exponent:

$$\int 12(2x-1)^{-2} dx$$

- Let $u = 2x - 1$. Then $du = 2dx$, or $dx = \frac{1}{2} du$.
- Substituting these into the integral:

$$\begin{aligned} \int 12u^{-2} \cdot \frac{1}{2} du &= 6 \int u^{-2} du \\ &= 6\left(\frac{u^{-1}}{-1}\right) + C \\ &= -\frac{6}{u} + C \end{aligned}$$

- Substituting back $u = 2x - 1$:

$$\boxed{-\frac{6}{2x-1} + C}$$

2. Partial Fraction Decomposition and Definite Integration

(a) Algebraic Rearrangement To write $\frac{4x+3}{x+2}$ in the form $A + \frac{B}{x+2}$, we perform **polynomial long division** or use algebraic manipulation:

$$\begin{aligned} \frac{4x+3}{x+2} &= \frac{4(x+2) - 8 + 3}{x+2} \\ &= \frac{4(x+2) - 5}{x+2} \\ &= 4 - \frac{5}{x+2} \end{aligned}$$

Thus, $A = 4$ and $B = -5$.

(b) Definite Integration We now evaluate the **definite integral** using the result from part (a):

$$I = \int_{-8}^{-5} \left(4 - \frac{5}{x+2}\right) dx$$

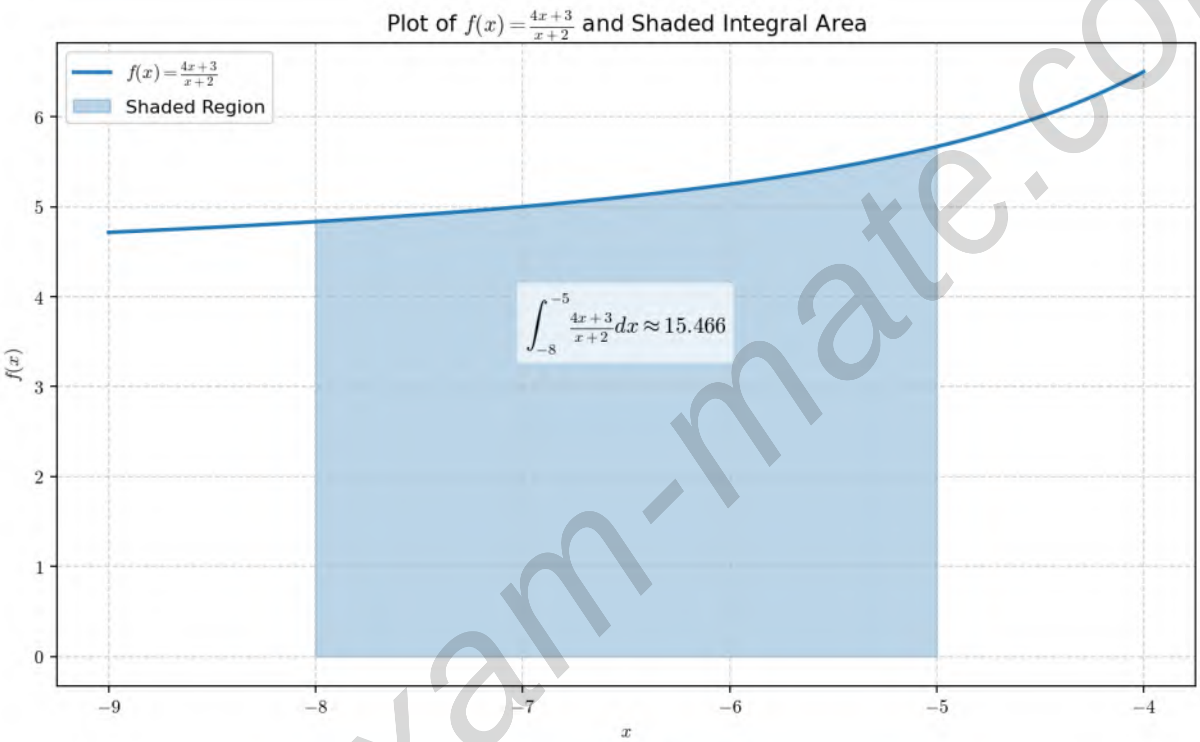
- The antiderivative of 4 is $4x$.
- The antiderivative of $\frac{5}{x+2}$ is $5 \ln|\ x + 2\ |$.

- Applying the **Fundamental Theorem of Calculus**:

$$\begin{aligned} I &= [4x - 5\ln|\ x+2\ |]_{-8}^{-5} \\ &= (4(-5) - 5\ln|-5+2\ |) - (4(-8) - 5\ln|-8+2\ |) \\ &= (-20 - 5\ln 3) - (-32 - 5\ln 6) \\ &= -20 - 5\ln 3 + 32 + 5\ln 6 \\ &= 12 + 5(\ln 6 - \ln 3) \end{aligned}$$

- Using the **logarithm quotient rule**, $\ln 6 - \ln 3 = \ln\left(\frac{6}{3}\right) = \ln 2$:

$$I = 12 + 5\ln 2$$



$$\boxed{12 + 5\ln 2}$$

WMA11_P3(IAL)_Summer_2021_Q4

Solution

1. Solving the composite equation $fg(x) = 3$

- First, we determine the expression for the **composite function** $f(g(x))$. Given $f(x) = \frac{4x+6}{x-5}$ and $g(x) = 5 - 2x^2$, we substitute $g(x)$ into f:

$$f(g(x)) = \frac{4(5-2x^2)+6}{(5-2x^2)-5}$$

- Simplify the expression:

$$\begin{aligned} f(g(x)) &= \frac{20-8x^2+6}{-2x^2} \\ &= \frac{26-8x^2}{-2x^2} \\ &= \frac{2(13-4x^2)}{-2x^2} \\ &= \frac{4x^2-13}{x^2} \end{aligned}$$

- Set the composite function equal to 3 and solve for x:

$$\begin{aligned} \frac{4x^2-13}{x^2} &= 3 \\ 4x^2-13 &= 3x^2 \\ x^2 &= 13 \\ x &= \pm\sqrt{13} \end{aligned}$$

- We must check the **domain** of $g(x)$, which is $x \leq 0$. Therefore, we reject the positive root.

$$\boxed{x = -\sqrt{13}}$$

2. Finding the inverse function f^{-1}

- To find the **inverse function**, let $y = f(x)$ and solve for x in terms of y:

$$y = \frac{4x+6}{x-5}$$

- Rearrange the equation:

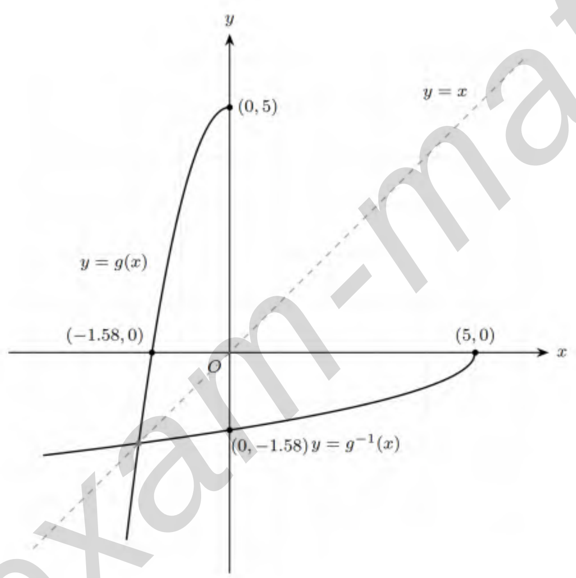
$$\begin{aligned} y(x-5) &= 4x+6 \\ yx-5y &= 4x+6 \\ yx-4x &= 5y+6 \\ x(y-4) &= 5y+6 \\ x &= \frac{5y+6}{y-4} \end{aligned}$$

- Replace x with $f^{-1}(x)$ and y with x:

$$\boxed{f^{-1}(x) = \frac{5x+6}{x-4}, \quad x \neq 4}$$

3. Sketching $y = g(x)$ and $y = g^{-1}(x)$

- The function $g(x) = 5 - 2x^2$ for $x \leq 0$ is the left half of a downward-opening parabola with vertex $(0, 5)$.
  - y-intercept: $g(0) = 5$. Point: $(0, 5)$.
  - x-intercept: $0 = 5 - 2x^2 \implies x^2 = 2.5 \implies x = -\sqrt{2.5} \approx -1.58$ (since $x \leq 0$). Point: $(-\sqrt{2.5}, 0)$.
- The inverse $g^{-1}(x)$ is the reflection of $g(x)$ in the line $y = x$.
  - The domain of g^{-1} is the range of g, which is $(-\infty, 5]$.
  - The y-intercept of g becomes the x-intercept of g^{-1}: $(5, 0)$.
  - The x-intercept of g becomes the y-intercept of g^{-1}: $(0, -\sqrt{2.5})$.



WMA11_P3(IAL)_Summer_2021_Q5

Solution

The growth of duckweed on a pond is modeled by the exponential equation:

$$A = pq^t$$

where A is the surface area in m^2, t is the time in days, and p, q are positive constants.

1. Linearization of the Model To relate the model to the provided graph of $\log_{10} A$ against t, we take the **base-10 logarithm** of both sides:

$$\begin{aligned}\log_{10} A &= \log_{10}(pq^t) \\ &= \log_{10} p + \log_{10}(q^t) \\ &= \log_{10} p + t \log_{10} q\end{aligned}$$

This is in the form of a linear equation $Y = mt + c$, where:

- The vertical intercept $c = \log_{10} p$
- The gradient $m = \log_{10} q$

2. Finding the values of p and q The line passes through the points $(0, 0.32)$ and $(8, 0.56)$.

- **Finding p:** From the intercept at $t = 0$:

$$\begin{aligned}\log_{10} p &= 0.32 \\ p &= 10^{0.32} \\ p &\approx 2.089296...\end{aligned}$$

  To 3 decimal places, $p = 2.089$.

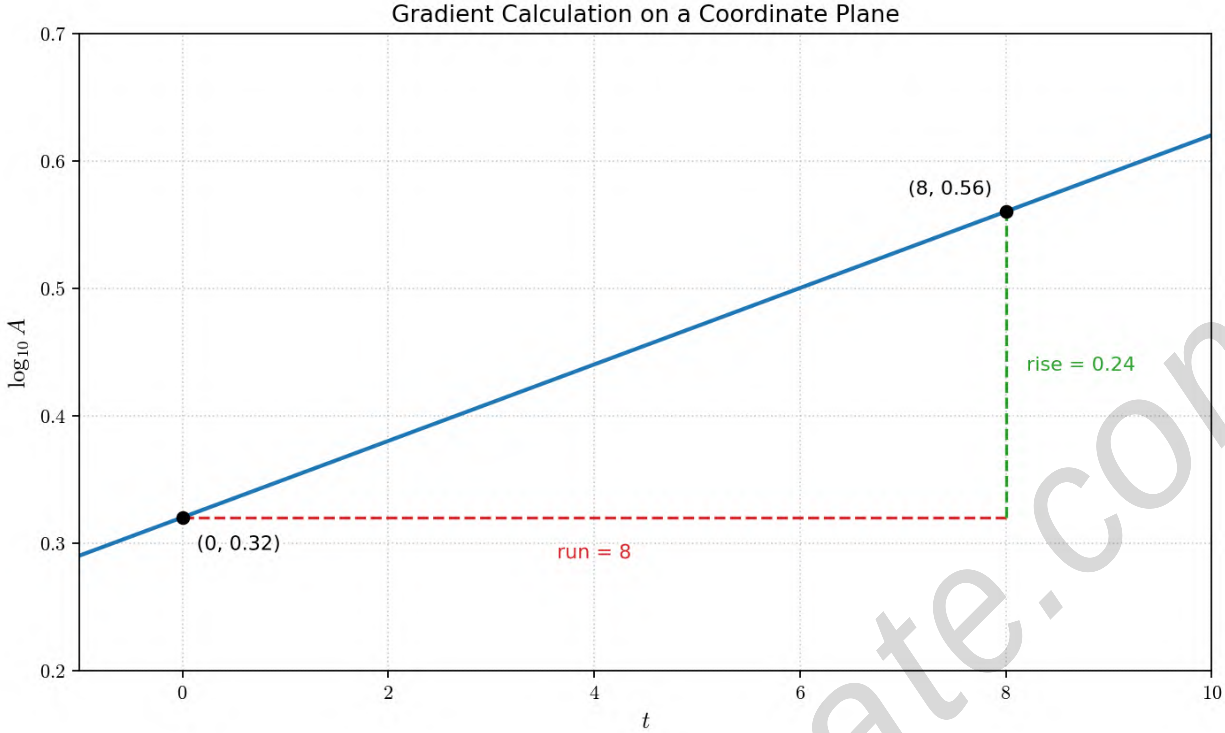
- **Finding q:** The gradient m is calculated as:

$$\begin{aligned}m &= \frac{0.56 - 0.32}{8 - 0} \\ &= \frac{0.24}{8} \\ &= 0.03\end{aligned}$$

  Since $m = \log_{10} q$:

$$\begin{aligned}\log_{10} q &= 0.03 \\ q &= 10^{0.03} \\ q &\approx 1.071519...\end{aligned}$$

  To 3 decimal places, $q = 1.072$.



3. Rate of increase of the surface area The **rate of increase** is given by the derivative dA/dt. Using the rule for differentiating exponential functions a^t:

$$A = pq^t$$
$$\frac{dA}{dt} = p(q^t \ln q)$$

We are asked for the rate at $t = 6$ days. Using the values $p = 10^{0.32}$ and $q = 10^{0.03}$:

$$\begin{aligned}
\left.\frac{dA}{dt}\right|_{t=6} &= 10^{0.32} \cdot (10^{0.03})^6 \cdot \ln(10^{0.03}) \\
&= 10^{0.32} \cdot 10^{0.18} \cdot \ln(10^{0.03}) \\
&= 10^{0.50} \cdot \ln(10^{0.03}) \\
&= \sqrt{10} \cdot (0.03 \ln 10) \\
&\approx 3.162277 \cdot 0.069077\ldots \\
&\approx 0.21844\ldots
\end{aligned}$$

Rounding to 2 decimal places, the rate is $0.22\,\mathrm{m}^2/\ \mathrm{day}$.

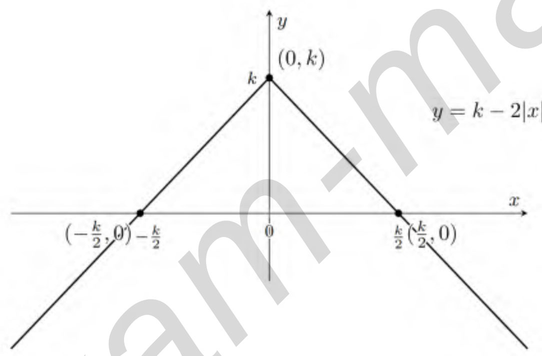
(a) $p = 2.089$, $q = 1.072$
(b) $0.22\,\mathrm{m}^2/\ \mathrm{day}$

WMA11_P3(IAL)_Summer_2021_Q6

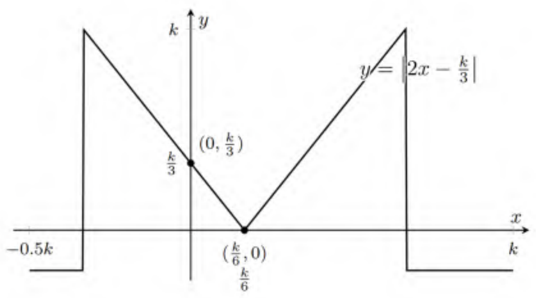
Solution

1. Sketching the graphs

- **(i) Graph of $y = k - 2 \mid x \mid$** The function $y = k - 2 \mid x \mid$ is an inverted V-shaped graph with its vertex at the y-axis.
  - **y-intercept**: Setting $x = 0$ gives $y = k - 2(0) = k$. The coordinate is $(0, k)$.
  - **x-intercepts**: Setting $y = 0$ gives $0 = k - 2 \mid x \mid$, which implies $\mid x \mid = \frac{k}{2}$. Thus, $x = \pm\frac{k}{2}$. The coordinates are $(\frac{k}{2}, 0)$ and $(-\frac{k}{2}, 0)$.



- **(ii) Graph of $y = \mid 2x - \frac{k}{3} \mid$** The function $y = \mid 2x - \frac{k}{3} \mid$ is a standard V-shaped **absolute value** graph reflected above the x-axis.
  - **y-intercept**: Setting $x = 0$ gives $y = \mid 2(0) - \frac{k}{3} \mid = \mid -\frac{k}{3} \mid = \frac{k}{3}$. The coordinate is $(0, \frac{k}{3})$.
  - **x-intercept**: Setting $y = 0$ gives $0 = \mid 2x - \frac{k}{3} \mid$, which implies $2x = \frac{k}{3}$, so $x = \frac{k}{6}$. The coordinate is $(\frac{k}{6}, 0)$.



2. Solving the equation $|2x - \frac{k}{3}| = k - 2|x|$

To find the values of x, we analyze the equation based on the critical points of the absolute value expressions, which are $x = 0$ and $x = \frac{k}{6}$. Since $k > 0$, we consider three intervals: $x < 0$, $0 \leq x < \frac{k}{6}$, and $x \geq \frac{k}{6}$.

- **Case 1: $x < 0$** In this region, $|x| = -x$ and $|2x - \frac{k}{3}| = -(2x - \frac{k}{3}) = -2x + \frac{k}{3}$.

$$-2x + \frac{k}{3} = k - 2(-x)$$
$$-2x + \frac{k}{3} = k + 2x$$
$$-4x = \frac{2k}{3}$$
$$x = -\frac{k}{6}$$

  Since $-\frac{k}{6} < 0$, this is a valid solution.

- **Case 2: $0 \leq x < \frac{k}{6}$** In this region, $|x| = x$ and $|2x - \frac{k}{3}| = -(2x - \frac{k}{3}) = -2x + \frac{k}{3}$.

$$-2x + \frac{k}{3} = k - 2x$$
$$\frac{k}{3} = k$$

  Since k is a positive constant, $\frac{k}{3} \neq k$. There is no solution in this interval.

- **Case 3: $x \geq \frac{k}{6}$** In this region, $|x| = x$ and $|2x - \frac{k}{3}| = 2x - \frac{k}{3}$.

$$2x - \frac{k}{3} = k - 2x$$
$$4x = \frac{4k}{3}$$
$$x = \frac{k}{3}$$

Since $\frac{k}{3} \geq \frac{k}{6}$, this is a valid solution.

The values of x are:

$$\boxed{x = -\frac{k}{6}, \quad x = \frac{k}{3}}$$

WMA11_P3(IAL)_Summer_2021_Q7

Solution

To find the derivative dy/dx and the integer constants A and B, we apply the principles of **implicit differentiation** and trigonometric substitution.

1. Differentiate the given expression with respect to y The given equation is:

$$x = 6\sin^2(2y)$$

Applying the **chain rule** to differentiate x with respect to y:

$$\begin{aligned}\frac{dx}{dy} &= 6 \cdot \frac{d}{dy}[(\sin(2y))^2] \\ &= 6 \cdot 2\sin(2y) \cdot \frac{d}{dy}[\sin(2y)] \\ &= 12\sin(2y) \cdot \cos(2y) \cdot 2 \\ &= 24\sin(2y)\cos(2y)\end{aligned}$$

Using the **double-angle formula** $\sin(2\theta) = 2\sin\theta\cos\theta$, we can simplify this expression:

$$\begin{aligned}\frac{dx}{dy} &= 12(2\sin(2y)\cos(2y)) \\ &= 12\sin(4y)\end{aligned}$$

2. Express the derivative in terms of x We need to find dy/dx, which is the reciprocal of dx/dy:

$$\frac{dy}{dx} = \frac{1}{24\sin(2y)\cos(2y)}$$

From the original equation, we have:

$$\sin^2(2y) = \frac{x}{6} \implies \sin(2y) = \sqrt{\frac{x}{6}}$$

Note that for $0 < y < \pi/4$, we have $0 < 2y < \pi/2$, so $\sin(2y)$ and $\cos(2y)$ are both positive. Using the **Pythagorean identity** $\cos^2\theta = 1 - \sin^2\theta$:

$$\begin{aligned}\cos(2y) &= \sqrt{1 - \sin^2(2y)} \\ &= \sqrt{1 - \frac{x}{6}} \\ &= \sqrt{\frac{6-x}{6}}\end{aligned}$$

3. Substitute back into the derivative formula Now, substitute the expressions for $\sin(2y)$ and $\cos(2y)$ into the derivative:

$$\frac{dy}{dx} = \frac{1}{24\left(\sqrt{\frac{x}{6}}\right)\left(\sqrt{\frac{6-x}{6}}\right)}$$
$$= \frac{1}{24\sqrt{\frac{x(6-x)}{36}}}$$
$$= \frac{1}{24 \cdot \frac{1}{6}\sqrt{6x - x^2}}$$
$$= \frac{1}{4\sqrt{6x - x^2}}$$

4. Identify the constants Comparing the result to the required form $\frac{dy}{dx} = \frac{1}{A\sqrt{Bx - x^2}}$:

- $A = 4$
- $B = 6$

Both A and B are integers as required.

$\boxed{A = 4, B = 6}$

WMA11_P3(IAL)_Summer_2021_Q8

Solution

The population of fish N at time t (in years) is modeled by the **logistic growth** function:

$$N = \frac{600e^{0.3t}}{2 + e^{0.3t}}, \quad t \geq 0$$

1. Initial population (Part a) To find the number of fish at the start of the study, we evaluate the function at $t = 0$:

$$\begin{aligned} N(0) &= \frac{600e^{0.3(0)}}{2 + e^{0.3(0)}} \\ &= \frac{600(1)}{2 + 1} \\ &= \frac{600}{3} \\ &= 200 \end{aligned}$$

The initial number of fish is $\boxed{200}$.

2. Upper limit of the population (Part b) The upper limit is found by taking the **limit** as $t \to \infty$. We divide the numerator and denominator by $e^{0.3t}$:

$$\begin{aligned} \lim_{t\to\infty} \frac{600e^{0.3t}}{2 + e^{0.3t}} &= \lim_{t\to\infty} \frac{600}{\frac{2}{e^{0.3t}} + 1} \\ &= \frac{600}{0 + 1} \\ &= 600 \end{aligned}$$

The upper limit to the number of fish is $\boxed{600}$.

3. Time to reach 500 fish (Part c) We set $N = 500$ and solve for t:

$$\begin{aligned} 500 &= \frac{600e^{0.3t}}{2 + e^{0.3t}} \\ 5(2 + e^{0.3t}) &= 6e^{0.3t} \\ 10 + 5e^{0.3t} &= 6e^{0.3t} \\ e^{0.3t} &= 10 \\ 0.3t &= \ln(10) \\ t &= \frac{\ln(10)}{0.3} \approx 7.67528... \text{ years} \end{aligned}$$

To convert the decimal part to months:

- $0.67528... \times 12 \approx 8.103... \text{ months}$ Rounding to the nearest month, we get 7 years and 8 months. The time is $\boxed{7 \text{ years and } 8 \text{ months}}$.

4. Derivative of the population model (Part d) To find dN/dt, we use the **quotient rule** where $u = 600e^{0.3t}$ and $v = 2 + e^{0.3t}$:

- $du/dt = 180e^{0.3t}$
- $dv/dt = 0.3e^{0.3t}$

$$\begin{aligned}\frac{dN}{dt} &= \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2} \\ &= \frac{(2+e^{0.3t})(180e^{0.3t}) - (600e^{0.3t})(0.3e^{0.3t})}{(2+e^{0.3t})^2} \\ &= \frac{360e^{0.3t} + 180e^{0.6t} - 180e^{0.6t}}{(2+e^{0.3t})^2} \\ &= \frac{360e^{0.3t}}{(2+e^{0.3t})^2}\end{aligned}$$

Comparing this to the form $\frac{Ae^{0.3t}}{(2+e^{0.3t})^2}$, we find $A = 360$.

5. Finding T when growth rate is 8 (Part e) Given $dN/dt = 8$ at $t = T$:

$$\frac{360e^{0.3T}}{(2+e^{0.3T})^2} = 8$$
$$45e^{0.3T} = (2+e^{0.3T})^2$$

Let $x = e^{0.3T}$:

$$45x = (2+x)^2$$
$$45x = 4 + 4x + x^2$$
$$x^2 - 41x + 4 = 0$$

Using the **quadratic formula**:

$$x = \frac{41 \pm \sqrt{41^2 - 4(1)(4)}}{2} = \frac{41 \pm \sqrt{1681-16}}{2} = \frac{41 \pm \sqrt{1665}}{2}$$

Calculating the two possible values for x:

- $x_1 \approx 40.902$
- $x_2 \approx 0.0978$ Since $t \geq 0$, $e^{0.3T} \geq 1$, so we take $x \approx 40.902$:

$$\begin{aligned}e^{0.3T} &= 40.9023... \\ 0.3T &= \ln(40.9023...) \\ T &= \frac{\ln(40.9023...)}{0.3} \approx 12.370...\end{aligned}$$

Rounding to one decimal place, we obtain $T = 12.4$. The value of T is $\boxed{12.4}$.

WMA11_P3(IAL)_Summer_2021_Q9

Solution

1. Harmonic Form Expression

To express $12\sin x - 5\cos x$ in the form $R\sin(x-\alpha)$, we use the **harmonic addition theorem**. Expanding the target form using the **sine subtraction formula**:

$$R\sin(x-\alpha) = R(\sin x\cos\alpha - \cos x\sin\alpha) = (R\cos\alpha)\sin x - (R\sin\alpha)\cos x$$

By comparing coefficients with $12\sin x - 5\cos x$:

- $R\cos\alpha = 12$
- $R\sin\alpha = 5$

To find the **amplitude** R:

$$\begin{aligned} R^2 &= 12^2 + 5^2 \\ &= 144 + 25 \\ &= 169 \\ R &= \sqrt{169} = 13 \end{aligned}$$

To find the phase angle α:

$$\begin{aligned} \tan\alpha &= \frac{R\sin\alpha}{R\cos\alpha} = \frac{5}{12} \\ \alpha &= \arctan\left(\frac{5}{12}\right) \\ &\approx 0.394791\ldots \end{aligned}$$

Rounding to 3 decimal places, we obtain $\alpha \approx 0.395\,\text{rad}$.

Thus, the expression is:

$$\boxed{13\sin(x-0.395)}$$

2. Analysis of the function $g(\theta)$

The function is defined as $g(\theta) = 10 + 12\sin\left(2\theta - \frac{\pi}{6}\right) - 5\cos\left(2\theta - \frac{\pi}{6}\right)$. Using the result from part (a) where $x = 2\theta - \frac{\pi}{6}$:

$$g(\theta) = 10 + 13\sin\left(2\theta - \frac{\pi}{6} - \alpha\right)$$

(i) Minimum value of $g(\theta)$ The minimum value of the **sine function** is -1.

$$\begin{aligned} g(\theta)_{\min} &= 10 + 13(-1) \\ &= 10 - 13 \\ &= -3 \end{aligned}$$

$\boxed{g(\theta)_{\min} = -3}$

(ii) Smallest value of $\theta > 0$ The minimum occurs when the argument of the sine function is $\frac{3\pi}{2}$ (or $-\frac{\pi}{2}$):

$$\begin{aligned} 2\theta - \frac{\pi}{6} - \alpha &= \frac{3\pi}{2} \\ 2\theta &= \frac{3\pi}{2} + \frac{\pi}{6} + \alpha \\ 2\theta &= \frac{5\pi}{3} + 0.394791... \\ 2\theta &= 5.235987... + 0.394791... \\ 2\theta &= 5.630779... \\ \theta &= 2.815389... \end{aligned}$$

Checking if there is a smaller positive θ by using the previous cycle ($-\frac{\pi}{2}$):

$$\begin{aligned} 2\theta - \frac{\pi}{6} - \alpha &= -\frac{\pi}{2} \\ 2\theta &= -\frac{\pi}{2} + \frac{\pi}{6} + \alpha \\ 2\theta &= -\frac{\pi}{3} + 0.394791... \\ 2\theta &= -1.047197... + 0.394791... \\ 2\theta &= -0.652406... \quad (\text{yields} \quad \theta < 0) \end{aligned}$$

Thus, the smallest positive value is: $\boxed{\theta \approx 2.815}$

3. Range of the function $h(\beta)$

The function is $h(\beta) = 10 - (12\sin\beta - 5\cos\beta)^2$. From part (a), we know $12\sin\beta - 5\cos\beta = 13\sin(\beta - \alpha)$. Substituting this into $h(\beta)$:

$$\begin{aligned} h(\beta) &= 10 - [13\sin(\beta - \alpha)]^2 \\ &= 10 - 169\sin^2(\beta - \alpha) \end{aligned}$$

The term $\sin^2(\beta - \alpha)$ oscillates between 0 and 1.

- When $\sin^2(\beta - \alpha) = 0$:

  $h(\beta) = 10 - 169(0) = 10$

- When $\sin^2(\beta - \alpha) = 1$:

  $h(\beta) = 10 - 169(1) = -159$

The **range** of the function is the interval between these two values.

$\boxed{-159 \leq h(\beta) \leq 10}$

WMA11_P3(IAL)_Winter_2021_Q1

Solution

To evaluate the indefinite integral $\int \frac{x^2-5}{2x^3}\,dx$ for $x > 0$, we simplify the integrand and apply the **Power Rule** for integration.

1. Simplify the integrand The expression can be split into two separate fractions by dividing each term in the numerator by the denominator:

$$\begin{aligned}\frac{x^2-5}{2x^3} &= \frac{x^2}{2x^3} - \frac{5}{2x^3} \\ &= \frac{1}{2x} - \frac{5}{2}x^{-3}\end{aligned}$$

2. Integrate term by term Using the **linearity property** of integrals, we integrate the two terms separately:

$$\begin{aligned}\int \left(\frac{1}{2x} - \frac{5}{2}x^{-3}\right) dx &= \int \frac{1}{2x}\,dx - \int \frac{5}{2}x^{-3}\,dx \\ &= \frac{1}{2}\int \frac{1}{x}\,dx - \frac{5}{2}\int x^{-3}\,dx\end{aligned}$$

3. Apply integration formulas

- For the first term, we use the rule $\int \frac{1}{x}\,dx = \ln|\ x\ | + C$. Since $x > 0$, we have $\ln(x)$.
- For the second term, we use the **Power Rule** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$:

$$\begin{aligned}\int x^{-3}\,dx &= \frac{x^{-3+1}}{-3+1} \\ &= \frac{x^{-2}}{-2} \\ &= -\frac{1}{2x^2}\end{aligned}$$

4. Combine the results Substitute the results back into the expression and include the **constant of integration** C:

$$\begin{aligned}I &= \frac{1}{2}\ln(x) - \frac{5}{2}\left(-\frac{1}{2x^2}\right) + C \\ &= \frac{1}{2}\ln(x) + \frac{5}{4x^2} + C\end{aligned}$$

$$\boxed{\frac{1}{2}\ln(x) + \frac{5}{4x^2} + C}$$

WMA11_P3(IAL)_Winter_2021_Q2

Solution

To solve for the transformations of the curve $y = f(x)$, we first identify the key points from the original graph:

- **y-intercept**: $(0, 0)$
- **Maximum turning point**: $(1, 2)$
- **Minimum turning point**: $(3, 0)$

---

1. Transformation (i): $y = 3f(2x)$

This equation involves two transformations:

- **Horizontal stretch** (compression) by a factor of $\frac{1}{2}$ due to the $2x$ term. Every x-coordinate is multiplied by 0.5.
- **Vertical stretch** by a factor of 3 due to the coefficient 3. Every y-coordinate is multiplied by 3.

Mapping the key points:

- **y-intercept**:

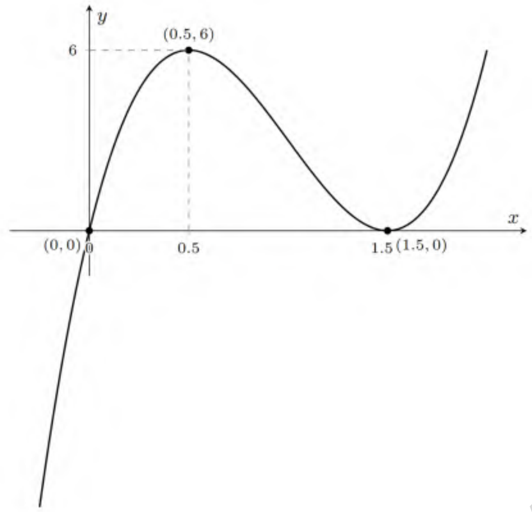
  $$(0, 0) \to (0 \cdot 0.5, 0 \cdot 3) = (0, 0)$$

- **Maximum turning point**:

  $$(1, 2) \to (1 \cdot 0.5, 2 \cdot 3) = (0.5, 6)$$

- **Minimum turning point**:

  $$(3, 0) \to (3 \cdot 0.5, 0 \cdot 3) = (1.5, 0)$$



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2. Transformation (ii): $y = f(-x) - 1$

This equation involves two transformations:

- **Reflection** in the y-axis due to the $-x$ term. The sign of every x-coordinate is reversed.
- **Vertical translation** downwards by 1 unit due to the -1 term. Every y-coordinate is decreased by 1.

Mapping the key points:

- **y-intercept**:

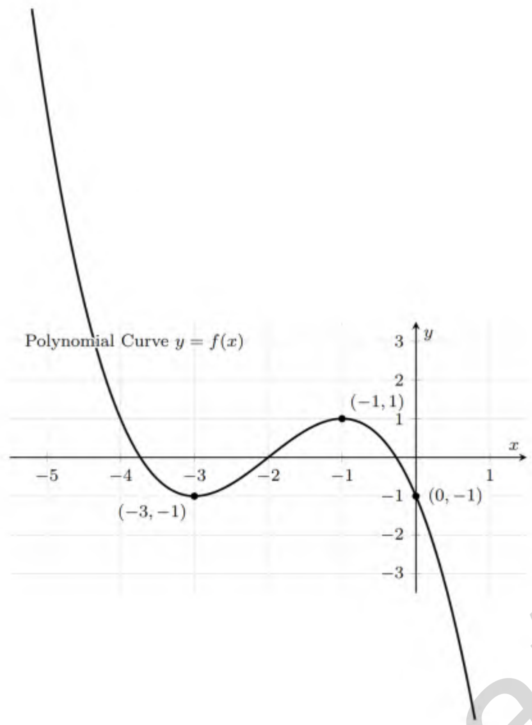
$$(0, 0) \to (0, 0 - 1) = (0, -1)$$

- **Maximum turning point**:

$$(1, 2) \to (-1, 2 - 1) = (-1, 1)$$

- **Minimum turning point**:

$$(3, 0) \to (-3, 0 - 1) = (-3, -1)$$



Summary of Coordinates

For $y = 3f(2x)$:

- y-intercept: $(0, 0)$
- Maximum: $(0.5, 6)$
- Minimum: $(1.5, 0)$

For $y = f(-x) - 1$:

- y-intercept: $(0, -1)$
- Maximum: $(-1, 1)$
- Minimum: $(-3, -1)$

WMA11_P3(IAL)_Winter_2021_Q3

Solution

1. Simplification of the Rational Function

To simplify the expression for $f(x)$, we first factor the denominator of the third term.

- Factorize the quadratic expression $2x^2 - 3x - 5$: We look for two numbers that multiply to $2 \times (-5) = -10$ and add to -3. These numbers are -5 and 2.

$$\begin{aligned} 2x^2 - 3x - 5 &= 2x^2 + 2x - 5x - 5 \\ &= 2x(x+1) - 5(x+1) \\ &= (2x-5)(x+1) \end{aligned}$$

- Express $f(x)$ with a **common denominator** of $(2x-5)(x+1)$:

$$\begin{aligned} f(x) &= 3 - \frac{x-2}{x+1} + \frac{5x+26}{(2x-5)(x+1)} \\ &= \frac{3(x+1)(2x-5) - (x-2)(2x-5) + (5x+26)}{(x+1)(2x-5)} \end{aligned}$$

- Expand the terms in the numerator:
  - $3(x+1)(2x-5) = 3(2x^2 - 3x - 5) = 6x^2 - 9x - 15$
  - $(x-2)(2x-5) = 2x^2 - 5x - 4x + 10 = 2x^2 - 9x + 10$
- Substitute these back into the numerator:

$$\begin{aligned} \text{Numerator} &= (6x^2 - 9x - 15) - (2x^2 - 9x + 10) + (5x + 26) \\ &= 6x^2 - 2x^2 - 9x + 9x + 5x - 15 - 10 + 26 \\ &= 4x^2 + 5x + 1 \end{aligned}$$

- Factor the resulting numerator $4x^2 + 5x + 1$:

$$4x^2 + 5x + 1 = (4x+1)(x+1)$$

- Simplify the fraction by canceling the common factor $(x+1)$:

$$\begin{aligned} f(x) &= \frac{(4x+1)(x+1)}{(2x-5)(x+1)} \\ &= \frac{4x+1}{2x-5} \end{aligned}$$

  Comparing this to the form $\frac{ax+b}{cx+d}$, we identify the integers: $a = 4, b = 1, c = 2, d = -5$.

2. Finding the Inverse Function

To find the **inverse function** $f^{-1}(x)$, we set $y = f(x)$ and solve for x:

$$y = \frac{4x+1}{2x-5}$$
$$y(2x-5) = 4x+1$$
$$2xy - 5y = 4x+1$$
$$2xy - 4x = 5y+1$$
$$x(2y-4) = 5y+1$$
$$x = \frac{5y+1}{2y-4}$$

Interchanging x and y gives:

$$f^{-1}(x) = \frac{5x+1}{2x-4}$$

3. Domain of the Inverse Function

The **domain** of f^{-1} is the **range** of f for the given domain $x > 4$.

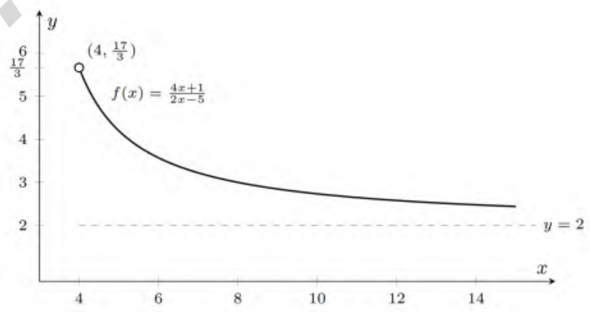
- As $x \to \infty$, $f(x) \to \frac{4}{2} = 2$.
- At the boundary $x = 4$:

$$f(4) = \frac{4(4)+1}{2(4)-5} = \frac{17}{3}$$

- Since $f(x)$ is a **homographic function** (a linear fractional transformation), we check its derivative to see if it is monotonic:

$$f'(x) = \frac{4(2x-5) - 2(4x+1)}{(2x-5)^2} = \frac{8x - 20 - 8x - 2}{(2x-5)^2} = \frac{-22}{(2x-5)^2}$$

Since $f'(x) < 0$ for all x in the domain, the function is strictly decreasing.



As x increases from 4 to ∞, $f(x)$ decreases from $\frac{17}{3}$ towards 2. Therefore, the range of f is $2 < f(x) < \frac{17}{3}$.

The domain of f^{-1} is:

$$\boxed{2 < x < \frac{17}{3}}$$

WMA11_P3(IAL)_Winter_2021_Q4

Solution

The problem involves analyzing two functions involving **absolute values** and finding their points of intersection. The functions are defined as:

$$f(x) = |\ 3x + a\ | + a$$
$$g(x) = |\ x + 5a\ |$$

where a is a positive constant.

1. Coordinates of the vertex P The **vertex** of a function of the form $y = |\ mx + c\ | + k$ occurs where the expression inside the absolute value is zero.

- For $f(x) = |\ 3x + a\ | + a$, we set the argument of the absolute value to zero:

$$3x + a = 0$$
$$x = -\frac{a}{3}$$

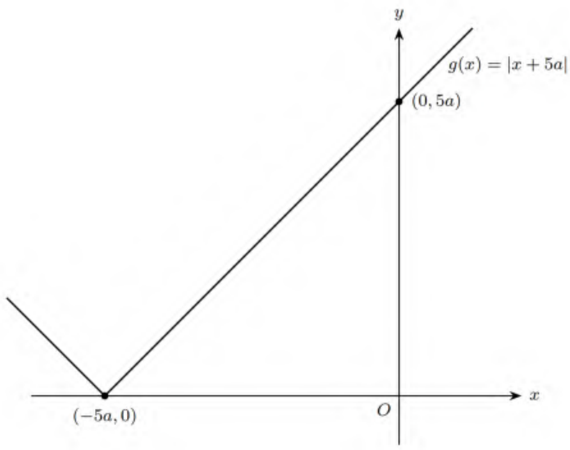
- Substituting $x = -\frac{a}{3}$ into $f(x)$:

$$f\left(-\frac{a}{3}\right) = |\ 3(-\frac{a}{3}) + a\ | + a$$
$$= |-a + a\ | + a$$
$$= a$$

  The coordinates of the vertex P are $\boxed{(-\frac{a}{3}, a)}$.

2. Sketch of $g(x) = |\ x + 5a\ |$ The graph of $g(x) = |\ x + 5a\ |$ is a V-shaped curve.

- **x-intercept**: Set $g(x) = 0 \implies |\ x + 5a\ | = 0 \implies x = -5a$. The point is $(-5a, 0)$.
- **y-intercept**: Set $x = 0 \implies g(0) = |\ 0 + 5a\ | = 5a$. The point is $(0, 5a)$.



3. Intersection points of $f(x)$ and $g(x)$ To find the intersection points, we solve the equation $f(x) = g(x)$:

$$|\ 3x + a\ | + a = |\ x + 5a\ |$$

Since $a > 0$, we consider the different regions defined by the critical points $x = -5a$ and $x = -\frac{a}{3}$.

Case 1: $x \geq -\frac{a}{3}$ In this region, $3x + a \geq 0$ and $x + 5a > 0$. The equation becomes:

$$(3x + a) + a = x + 5a$$
$$3x + 2a = x + 5a$$
$$2x = 3a$$
$$x = \frac{3}{2}a$$

To find the y-coordinate: $y = g(\frac{3}{2}a) = |\ \frac{3}{2}a + 5a\ | = \frac{13}{2}a$. The first intersection point is $(\frac{3}{2}a, \frac{13}{2}a)$.

Case 2: $-5a \leq x < -\frac{a}{3}$ In this region, $3x + a < 0$ and $x + 5a \geq 0$. The equation becomes:

$$-(3x + a) + a = x + 5a$$
$$-3x - a + a = x + 5a$$
$$-3x = x + 5a$$
$$-4x = 5a$$
$$x = -\frac{5}{4}a$$

Check if $-\frac{5}{4}a$ is in the interval $[-5a, -\frac{a}{3}]$: Since $-5 < -1.25 < -0.333$, the value $x = -1.25a$ is valid. To find the y-coordinate: $y = g(-\frac{5}{4}a) = |-\frac{5}{4}a + 5a\ | = \frac{15}{4}a$. The second intersection point is $(-\frac{5}{4}a, \frac{15}{4}a)$.

Case 3: $x < -5a$ In this region, both arguments are negative:

$$\begin{aligned}-(3x+a)+a &= -(x+5a)\\ -3x &= -x-5a\\ -2x &= -5a\\ x &= \frac{5}{2}a\end{aligned}$$

This contradicts the condition $x < -5a$ (since $a > 0$), so there are no solutions in this region.

The coordinates of the two points of intersection are:

$$\boxed{(\frac{3}{2}a, \frac{13}{2}a) \text{ and } (-\frac{5}{4}a, \frac{15}{4}a)}$$

WMA11_P3(IAL)_Winter_2021_Q5

Solution

The temperature θ in degrees Celsius ($^\circ$ C) at time t minutes is modeled by the **exponential decay** function:

$$\theta = A - 180e^{-kt}$$

1. Finding the value of A

- The initial temperature is given as 18° C at $t = 0$.
- Substituting these values into the equation:

$$18 = A - 180e^{-k(0)}$$
$$18 = A - 180(1)$$
$$A = 18 + 180$$
$$A = 198$$

$\boxed{A = 198}$

2. Showing the form of k

- Given that $\theta = 90^\circ$ C when $t = 5$ minutes:

$$90 = 198 - 180e^{-5k}$$
$$180e^{-5k} = 198 - 90$$
$$180e^{-5k} = 108$$
$$e^{-5k} = \frac{108}{180}$$
$$e^{-5k} = 0.6$$

- Taking the **natural logarithm** ($\ln$) of both sides:

$$-5k = \ln(0.6)$$
$$k = -\frac{1}{5}\ln\left(\frac{3}{5}\right)$$
$$k = \frac{1}{5}\ln\left(\left(\frac{3}{5}\right)^{-1}\right)$$
$$k = \frac{1}{5}\ln\left(\frac{5}{3}\right)$$

- Comparing this to the form $k = p\ln q$, we find $p = \frac{1}{5}$ and $q = \frac{5}{3}$. $\boxed{k = \frac{1}{5}\ln\left(\frac{5}{3}\right)}$

3. Temperature after 9 minutes

- Substitute $t = 9$ and the expression for k into the original equation:

$$\begin{aligned}\theta &= 198 - 180e^{-9\cdot\frac{1}{5}\ln(5/3)} \\ &= 198 - 180e^{\ln((5/3)^{-9/5})} \\ &= 198 - 180\left(\frac{5}{3}\right)^{-1.8} \\ &= 198 - 180\left(\frac{3}{5}\right)^{1.8}\end{aligned}$$

- Calculating the numerical value:

$$\begin{aligned}\theta &\approx 198 - 180(0.3986...) \\ &\approx 198 - 71.75... \\ &\approx 126.24...\end{aligned}$$

- To 3 significant figures, the temperature is 126° C. $\boxed{126^\circ \text{ C}}$

4. Rate of increase of temperature after 9 minutes

- The rate of increase is given by the **derivative** $d\theta/dt$:

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d}{dt}(198 - 180e^{-kt}) \\ &= 180ke^{-kt}\end{aligned}$$

- Substitute $t = 9$ and $k = \frac{1}{5}\ln(5/3) \approx 0.102165$:

$$\begin{aligned}\frac{d\theta}{dt} &= 180 \cdot \left(\frac{1}{5}\ln\left(\frac{5}{3}\right)\right) \cdot e^{-9\cdot\frac{1}{5}\ln(5/3)} \\ &= 36\ln\left(\frac{5}{3}\right) \cdot \left(\frac{3}{5}\right)^{1.8} \\ &\approx 18.3897... \cdot 0.3986... \\ &\approx 7.331...\end{aligned}$$

- To 3 significant figures, the rate is $7.33^\circ\ \text{C}\cdot\text{min}^{-1}$. $\boxed{7.33^\circ\ \text{C}\cdot\text{min}^{-1}}$

WMA11_P3(IAL)_Winter_2021_Q6

Solution

1. Differentiation of $f(x)$

To find $f'(x)$ for the function $f(x) = x\cos\left(\frac{x}{3}\right)$, we apply the **product rule**, which states that $\frac{d}{dx}[uv] = u'v + uv'$. Let $u = x$ and $v = \cos\left(\frac{x}{3}\right)$.

- The derivative of u is $u' = 1$.
- To find the derivative of v, we use the **chain rule**: $v' = -\sin\left(\frac{x}{3}\right) \cdot \frac{1}{3}$.

Substituting these into the product rule formula:

$$\begin{aligned} f'(x) &= (1) \cdot \cos\left(\frac{x}{3}\right) + x \cdot \left[-\frac{1}{3}\sin\left(\frac{x}{3}\right)\right] \\ &= \cos\left(\frac{x}{3}\right) - \frac{x}{3}\sin\left(\frac{x}{3}\right) \end{aligned}$$

$$\boxed{f'(x) = \cos\left(\frac{x}{3}\right) - \frac{x}{3}\sin\left(\frac{x}{3}\right)}$$

2. Rearranging $f'(x) = 0$

We set the derivative equal to zero and solve for x:

$$\begin{aligned} \cos\left(\frac{x}{3}\right) - \frac{x}{3}\sin\left(\frac{x}{3}\right) &= 0 \\ \cos\left(\frac{x}{3}\right) &= \frac{x}{3}\sin\left(\frac{x}{3}\right) \\ \frac{3}{x} &= \frac{\sin(x/3)}{\cos(x/3)} \\ \tan\left(\frac{x}{3}\right) &= \frac{3}{x} \end{aligned}$$

Taking the **arctangent** of both sides:

$$\begin{aligned} \frac{x}{3} &= \arctan\left(\frac{3}{x}\right) \\ x &= 3\arctan\left(\frac{3}{x}\right) \end{aligned}$$

Comparing this to the form $x = k\arctan\left(\frac{k}{x}\right)$, we find that the integer constant is $k = 3$.

$\boxed{k = 3}$

3. Iterative Calculation

Using the **fixed-point iteration** formula $x_{n+1} = 3\arctan\left(\frac{3}{x_n}\right)$ with the starting value $x_1 = 2.5$:

- For x_2:

$$x_2 = 3\arctan\left(\frac{3}{2.5}\right) \approx 2.62848\ldots$$

- Continuing the iterations:
  - $x_3 = 3\arctan\left(\frac{3}{2.62848}\right) \approx 2.55349...$
  - $x_4 = 3\arctan\left(\frac{3}{2.55349}\right) \approx 2.59784...$
  - $x_5 = 3\arctan\left(\frac{3}{2.59784}\right) \approx 2.57140...$
  - $x_6 = 3\arctan\left(\frac{3}{2.57140}\right) \approx 2.58706...$

Rounding to 3 decimal places: $\boxed{x_2 = 2.628, \quad x_6 = 2.587}$

4. Root Verification

To show that a root exists at $x = 2.581$ correct to 3 decimal places, we use the **Intermediate Value Theorem**. We define a function $g(x) = f'(x) = \cos\left(\frac{x}{3}\right) - \frac{x}{3}\sin\left(\frac{x}{3}\right)$ and test the boundaries of the interval $[2.5805, 2.5815]$.

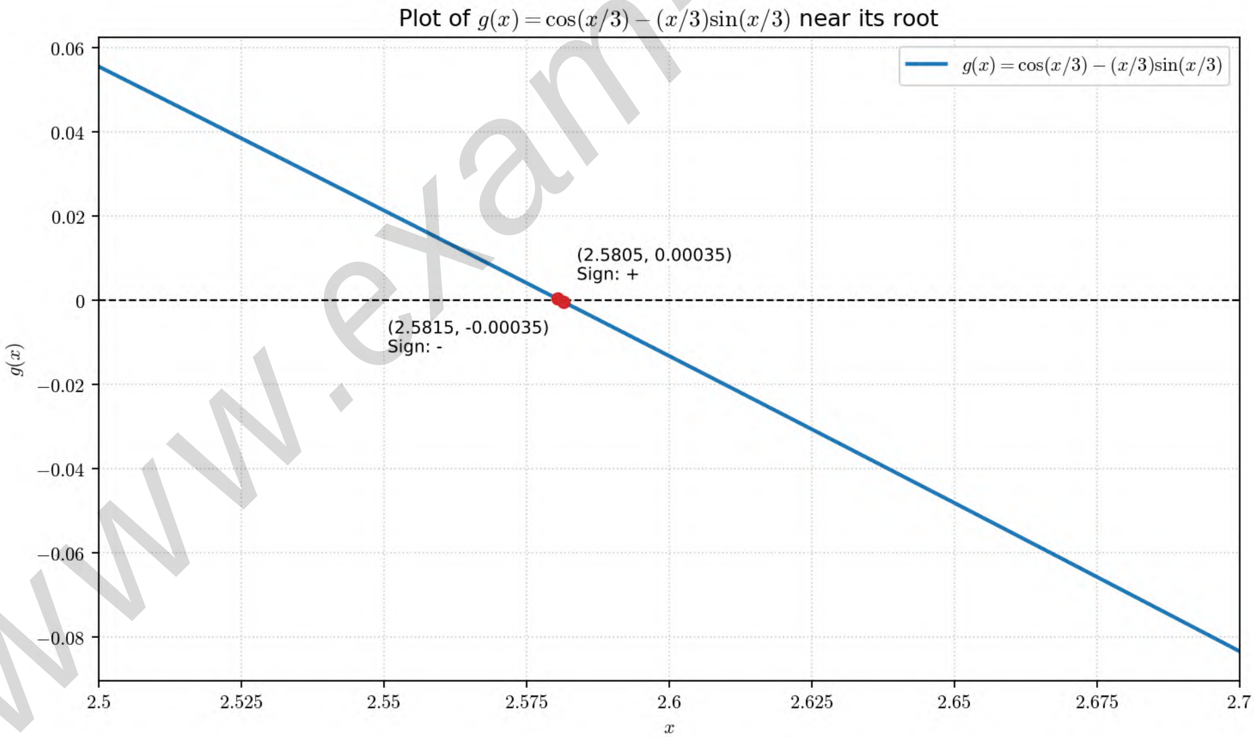
- At $x = 2.5805$:

$$g(2.5805) = \cos\left(\frac{2.5805}{3}\right) - \frac{2.5805}{3}\sin\left(\frac{2.5805}{3}\right) \approx 0.00013...$$

- At $x = 2.5815$:

$$g(2.5815) = \cos\left(\frac{2.5815}{3}\right) - \frac{2.5815}{3}\sin\left(\frac{2.5815}{3}\right) \approx -0.00031...$$

Since $g(2.5805) > 0$ and $g(2.5815) < 0$, there is a **sign change** in the continuous function $g(x)$ over the interval. Therefore, a root α exists such that $2.5805 < \alpha < 2.5815$, which rounds to 2.581 to 3 decimal places.



WMA11_P3(IAL)_Winter_2021_Q7

Solution

1. Proof of the Trigonometric Identity

To prove the identity $\frac{\sin 2x}{\cos x} + \frac{\cos 2x}{\sin x} \equiv \csc x$, we start by simplifying the left-hand side (LHS) using **double-angle formulas**.

- **Step 1: Expand the double-angle terms.** Recall that $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 1 - 2\sin^2 x$.

$$\text{LHS} = \frac{2\sin x\cos x}{\cos x} + \frac{1 - 2\sin^2 x}{\sin x}$$

- **Step 2: Simplify the fractions.** Assuming $x \neq \frac{n\pi}{2}$ (so $\sin x \neq 0$ and $\cos x \neq 0$), we can cancel $\cos x$ in the first term and split the second term:

$$\begin{aligned}\text{LHS} &= 2\sin x + \left(\frac{1}{\sin x} - \frac{2\sin^2 x}{\sin x}\right) \\ &= 2\sin x + \frac{1}{\sin x} - 2\sin x \\ &= \frac{1}{\sin x}\end{aligned}$$

- **Step 3: Relate to the right-hand side.** By the definition of the **cosecant** function:

$$\text{LHS} = \csc x = \text{RHS}$$

The identity is proven for $x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$.

2. Solving the Equation

We are given the equation:

$$7 + \frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta} = 3\cot^2 2\theta$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

- **Step 1: Apply the identity from part (a).** Let $x = 2\theta$. The expression $\frac{\sin 4\theta}{\cos 2\theta} + \frac{\cos 4\theta}{\sin 2\theta}$ matches the LHS of the identity proven in part (a). Thus:

$$7 + \csc 2\theta = 3\cot^2 2\theta$$

- **Step 2: Convert to a single trigonometric function.** Using the **Pythagorean identity** $1 + \cot^2 A = \csc^2 A$, we substitute $\cot^2 2\theta = \csc^2 2\theta - 1$:

$$\begin{aligned}7 + \csc 2\theta &= 3(\csc^2 2\theta - 1) \\ 7 + \csc 2\theta &= 3\csc^2 2\theta - 3 \\ 3\csc^2 2\theta - \csc 2\theta - 10 &= 0\end{aligned}$$

- **Step 3: Solve the quadratic equation.** Let $u = \csc 2\theta$. The equation becomes $3u^2 - u - 10 = 0$. Factoring the quadratic:

$$(3u+5)(u-2)=0$$

This gives two possible values for $\csc 2\theta$:

1. $\csc 2\theta = 2 \Longrightarrow \sin 2\theta = \frac{1}{2}$
2. $\csc 2\theta = -\frac{5}{3} \Longrightarrow \sin 2\theta = -0.6$

- **Step 4: Find the values of θ in the given range.** The range for θ is $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so the range for 2θ is $-\pi < 2\theta < \pi$.

Case 1: $\sin 2\theta = 0.5$

$$2\theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$$
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

In decimals (3 s.f.): $\theta \approx 0.262, 1.31$.

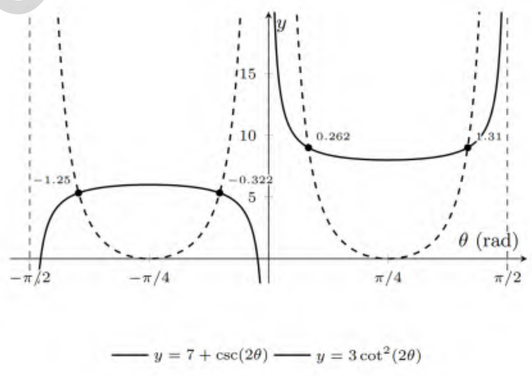
Case 2: $\sin 2\theta = -0.6$ Using the inverse sine function:

$$2\theta = \arcsin(-0.6) \approx -0.6435$$
$$2\theta = -\pi - \arcsin(-0.6) \quad \text{(outside range)}$$
$$2\theta = \pi - (-0.6435) = 3.785 \quad \text{(outside range)}$$

The other solution within $(-\pi, \pi)$ is:

$$2\theta = -\pi + 0.6435 = -2.498$$
$$\theta = -0.32175, -1.249$$

In decimals (3 s.f.): $\theta \approx -0.322, -1.25$.



The solutions for θ are:

$$\boxed{\theta = -1.25, -0.322, 0.262, 1.31}$$

WMA11_P3(IAL)_Winter_2021_Q8

Solution

1. Linearization of the Exponential Model

The percentage of the population with internet access, P, is modeled by the **exponential equation**:

$$P = ab^t$$

where t is the number of years after the start of 2005. To relate this to the given linear graph of $\log_{10} P$ against t, we apply the **logarithm** to both sides:

$$\begin{aligned}\log_{10} P &= \log_{10}(ab^t) \\ &= \log_{10} a + \log_{10}(b^t) \\ &= \log_{10} a + t \log_{10} b\end{aligned}$$

This equation is in the form of a **straight line** $y = mx + c$, where:

- $y = \log_{10} P$
- $x = t$
- $m = \log_{10} b$ (the gradient)
- $c = \log_{10} a$ (the intercept on the $\log_{10} P$ axis)

2. Finding the values of a and b

From the problem, the gradient $m = 0.09$ and the intercept $c = 0.68$.

- **To find a:**

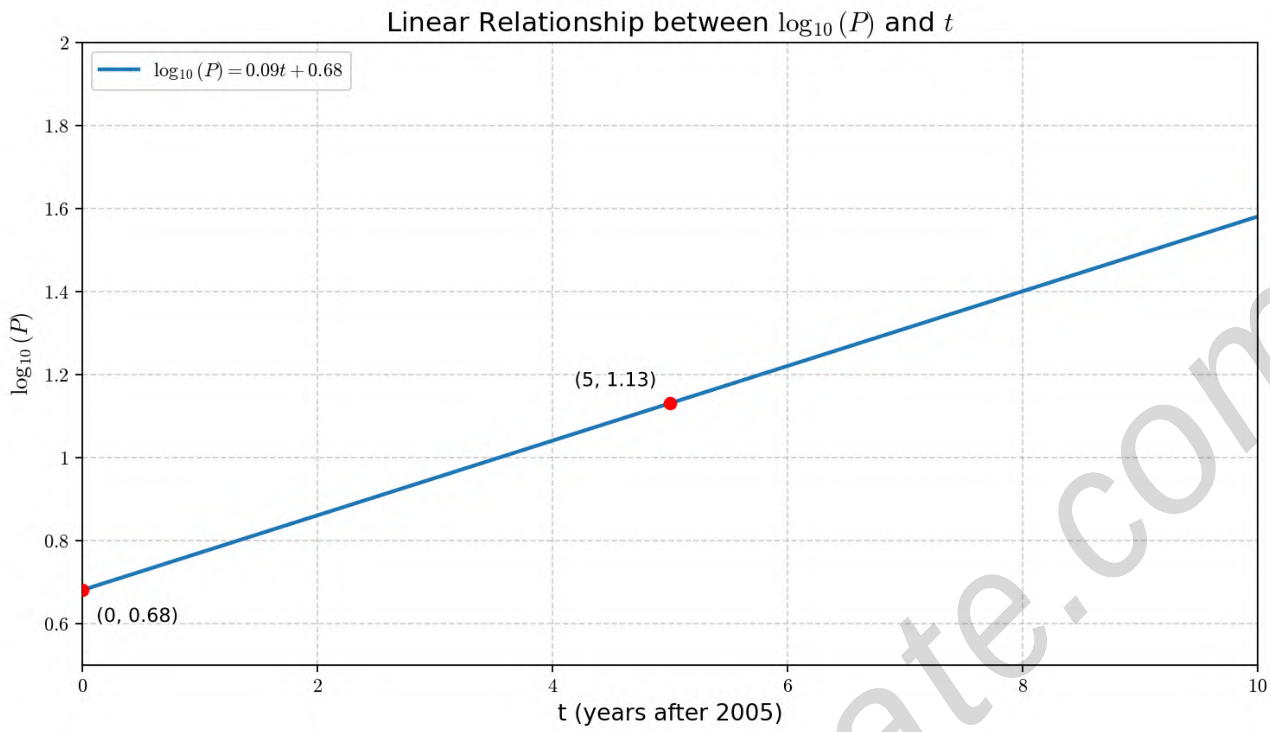
$$\begin{aligned}\log_{10} a &= 0.68 \\ a &= 10^{0.68} \\ a &\approx 4.7863\end{aligned}$$

  Rounding to 2 decimal places, $a = 4.79$.

- **To find b:**

$$\begin{aligned}\log_{10} b &= 0.09 \\ b &= 10^{0.09} \\ b &\approx 1.2302\end{aligned}$$

  Rounding to 2 decimal places, $b = 1.23$.



3. Practical interpretation of the constant a

In the context of the model, a represents the value of P when $t = 0$. Since t is the number of years after the start of 2005, $t = 0$ corresponds to the start of 2005. Therefore, a is the percentage of the population who had access to the internet at the start of 2005.

4. Estimating internet access at the start of 2015

The start of 2015 corresponds to $t = 2015 - 2005 = 10$ years. Using the model $P = ab^t$ with the unrounded values for better precision:

$$\begin{aligned} P &= 10^{0.68} \cdot (10^{0.09})^{10} \\ &= 10^{0.68} \cdot 10^{0.9} \\ &= 10^{0.68+0.9} \\ &= 10^{1.58} \\ P &\approx 38.0189 \end{aligned}$$

The percentage of the population with internet access at the start of 2015 is approximately 38.0%.

Final Answers:

(a) $\boxed{a = 4.79, b = 1.23}$

(b) **The percentage of the population with internet access at the start of 2005.**

(c) $\boxed{38.0\%}$

WMA11_P3(IAL)_Winter_2021_Q9

Solution

1. Evaluation of the first integral

To evaluate the integral $I_1 = \int \frac{3x-2}{3x^2-4x+5}\,dx$, we first examine the relationship between the numerator and the derivative of the denominator.

- Let the denominator be $f(x) = 3x^2 - 4x + 5$.
- The derivative is $f'(x) = \frac{d}{dx}(3x^2 - 4x + 5) = 6x - 4$.

We observe that the numerator $3x - 2$ is exactly half of the derivative of the denominator:

$$3x - 2 = \frac{1}{2}(6x - 4)$$

Using the **substitution method**, let $u = 3x^2 - 4x + 5$. Then, the differential is:

$$du = (6x-4)\,dx = 2(3x-2)\,dx \implies (3x-2)\,dx = \frac{1}{2}\,du$$

Substituting these into the integral:

$$\begin{aligned} I_1 &= \int \frac{1}{u} \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int \frac{1}{u}\,du \\ &= \frac{1}{2} \ln|\ u\ | + C \end{aligned}$$

Substituting back $u = 3x^2 - 4x + 5$:

$$I_1 = \frac{1}{2} \ln|\ 3x^2 - 4x + 5\ | + C$$

Since the **discriminant** of the quadratic $3x^2 - 4x + 5$ is $\Delta = (-4)^2 - 4(3)(5) = 16 - 60 = -44 < 0$ and the leading coefficient is positive, the expression $3x^2 - 4x + 5$ is always positive for all real x. Thus, the absolute value signs can be replaced with parentheses.

$$\boxed{\frac{1}{2} \ln(3x^2 - 4x + 5) + C}$$

2. Evaluation of the second integral

To evaluate the integral $I_2 = \int \frac{e^{2x}}{(e^{2x}-1)^3}\,dx$ for $x \neq 0$, we use the **u-substitution** technique.

- Let $u = e^{2x} - 1$.
- Differentiating with respect to x gives $\frac{du}{dx} = 2e^{2x}$, which implies $e^{2x}\,dx = \frac{1}{2}\,du$.

Substituting these into the integral expression:

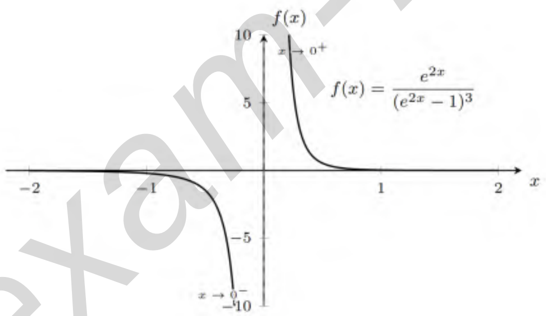
$$\begin{aligned} I_2 &= \int \frac{1}{u^3} \cdot \frac{1}{2}\, du \\ &= \frac{1}{2} \int u^{-3}\, du \end{aligned}$$

Applying the **power rule for integration** $\int u^n\, du = \frac{u^{n+1}}{n+1} + C$:

$$\begin{aligned} I_2 &= \frac{1}{2}\left[\frac{u^{-3+1}}{-3+1}\right] + C \\ &= \frac{1}{2}\left[\frac{u^{-2}}{-2}\right] + C \\ &= -\frac{1}{4u^2} + C \end{aligned}$$

Finally, substitute back $u = e^{2x} - 1$ to obtain the result in terms of x:

$$I_2 = -\frac{1}{4(e^{2x}-1)^2} + C$$



$$\boxed{-\frac{1}{4(e^{2x}-1)^2} + C}$$

WMA11_P3(IAL)_Winter_2021_Q10

Solution

1. Differentiation of x with respect to y

The curve C is defined by the equation $x = 3\sec^2(2y)$ for $x > 3$ and $0 < y < \frac{\pi}{4}$. To find $\frac{dx}{dy}$, we apply the **chain rule** to the function $x = 3[\sec(2y)]^2$.

- Let $u = \sec(2y)$. Then $x = 3u^2$.
- The derivative of u with respect to y is $\frac{du}{dy} = \sec(2y)\tan(2y) \cdot 2$.
- Applying the chain rule:

$$\begin{aligned}\frac{dx}{dy} &= 3 \cdot 2\sec(2y) \cdot \frac{d}{dy}(\sec(2y)) \\ &= 6\sec(2y) \cdot [2\sec(2y)\tan(2y)] \\ &= 12\sec^2(2y)\tan(2y)\end{aligned}$$

$$\boxed{\frac{dx}{dy} = 12\sec^2(2y)\tan(2y)}$$

2. Expressing $\frac{dy}{dx}$ in terms of x

To show the required form, we use the identity $\frac{dy}{dx} = \frac{1}{dx/dy}$ and convert the trigonometric terms into algebraic terms involving x.

- From the original equation, $\sec^2(2y) = \frac{x}{3}$.
- Using the **Pythagorean identity** $\tan^2(\theta) = \sec^2(\theta) - 1$:

$$\begin{aligned}\tan(2y) &= \sqrt{\sec^2(2y) - 1} \\ &= \sqrt{\frac{x}{3} - 1} \\ &= \sqrt{\frac{x-3}{3}} = \frac{\sqrt{x-3}}{\sqrt{3}}\end{aligned}$$

- Substitute these into the expression for $\frac{dx}{dy}$:

$$\begin{aligned}\frac{dx}{dy} &= 12\left(\frac{x}{3}\right)\left(\frac{\sqrt{x-3}}{\sqrt{3}}\right) \\ &= \frac{4x\sqrt{x-3}}{\sqrt{3}}\end{aligned}$$

- Therefore, the derivative $\frac{dy}{dx}$ is:

$$\frac{dy}{dx} = \frac{\sqrt{3}}{4x\sqrt{x-3}}$$

Comparing this to the form $\frac{p}{qx\sqrt{x-3}}$, we identify $p = \sqrt{3}$ and $q = 4$.

$$\boxed{p = \sqrt{3}, q = 4}$$

3. Equation of the normal at $y = \frac{\pi}{12}$

To find the **normal equation**, we first determine the coordinates (x, y) and the gradient of the tangent at the given point.

- **Find x at $y = \frac{\pi}{12}$:**

$$x = 3\sec^2\left(2 \cdot \frac{\pi}{12}\right) = 3\sec^2\left(\frac{\pi}{6}\right)$$
$$= 3 \cdot \left(\frac{2}{\sqrt{3}}\right)^2 = 3 \cdot \frac{4}{3} = 4$$

- **Find the gradient of the tangent (m_t):** Substitute $x = 4$ into the expression for $\frac{dy}{dx}$:

$$m_t = \frac{\sqrt{3}}{4(4)\sqrt{4-3}} = \frac{\sqrt{3}}{16}$$

- **Find the gradient of the normal (m):** The normal is perpendicular to the tangent, so $m = -\frac{1}{m_t}$:

$$m = -\frac{16}{\sqrt{3}} = -\frac{16\sqrt{3}}{3}$$

- **Find the equation of the normal:** Using the point-slope form $y - y_1 = m(x - x_1)$ with $(4, \frac{\pi}{12})$:

$$y - \frac{\pi}{12} = -\frac{16\sqrt{3}}{3}(x-4)$$
$$y = -\frac{16\sqrt{3}}{3}x + \frac{64\sqrt{3}}{3} + \frac{\pi}{12}$$

  The equation is in the form $y = mx + c$, where $m = -\frac{16\sqrt{3}}{3}$ and $c = \frac{64\sqrt{3}}{3} + \frac{\pi}{12}$.

$$\boxed{y = -\frac{16\sqrt{3}}{3}x + \left(\frac{64\sqrt{3}}{3} + \frac{\pi}{12}\right)}$$

WMA11_P3(IAL)_Summer_2022_Q1

Solution

1. Differentiation of the curve equation

To find the derivative of the function $y = (3x-2)^6$, we apply the **Chain Rule**. The chain rule states that for a composite function $y = f(g(x))$, the derivative is given by $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$.

- Let $u = 3x - 2$, then $y = u^6$.
- The derivative of y with respect to u is $\frac{dy}{du} = 6u^5$.
- The derivative of u with respect to x is $\frac{du}{dx} = 3$.

Combining these using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 6(3x-2)^5 \cdot 3 \\ &= 18(3x-2)^5\end{aligned}$$

$$\boxed{\frac{dy}{dx} = 18(3x-2)^5}$$

2. Finding the equation of the normal at $P\left(\frac{1}{3}, 1\right)$

- **Step 2.1: Calculate the gradient of the tangent at P** Substitute $x = \frac{1}{3}$ into the derivative expression:

$$\begin{aligned}m_{\text{tangent}} &= \frac{dy}{dx}|_{x=1/3} \\ &= 18\left(3\left(\frac{1}{3}\right) - 2\right)^5 \\ &= 18(1-2)^5 \\ &= 18(-1)^5 \\ &= -18\end{aligned}$$

- **Step 2.2: Calculate the gradient of the normal** The **normal line** is perpendicular to the tangent line. Therefore, the product of their gradients is -1:

$$\begin{aligned}m_{\text{normal}} &= -\frac{1}{m_{\text{tangent}}} \\ &= -\frac{1}{-18} \\ &= \frac{1}{18}\end{aligned}$$

- **Step 2.3: Determine the equation of the normal** Using the point-slope form of a linear equation, $y - y_1 = m(x - x_1)$, with point $P\left(\frac{1}{3}, 1\right)$ and $m = \frac{1}{18}$:

$$y - 1 = \frac{1}{18}\left(x - \frac{1}{3}\right)$$

Multiply the entire equation by 18 to clear the fraction on the right:

$$18(y-1) = x - \frac{1}{3}$$

$$18y - 18 = x - \frac{1}{3}$$

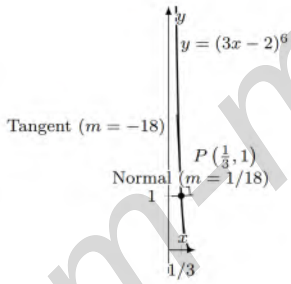
Multiply by 3 to eliminate the remaining fraction:

$$54y - 54 = 3x - 1$$

Rearrange into the general form $ax + by + c = 0$:

$$0 = 3x - 54y - 1 + 54$$

$$3x - 54y + 53 = 0$$



$$\boxed{3x - 54y + 53 = 0}$$

WMA11_P3(IAL)_Summer_2022_Q2

Solution

The functions f and g are defined as follows:

$$f(x) = \frac{5-x}{3x+2}, \quad x \in \mathbb{R}, x \neq -\frac{2}{3}$$

$$g(x) = 2x - 7, \quad x \in \mathbb{R}$$

1. Evaluation of the composite function $fg(5)$ To find the value of the **composite function** $fg(5)$, we first evaluate the inner function $g(x)$ at $x = 5$:

- $g(5) = 2(5) - 7 = 10 - 7 = 3$ Next, we substitute this result into the function $f(x)$:
- $f(g(5)) = f(3) = \frac{5-3}{3(3)+2} = \frac{2}{9+2} = \frac{2}{11}$

$\boxed{fg(5) = \frac{2}{11}}$

2. Determination of the inverse function f^{-1} To find the **inverse function** $f^{-1}(x)$, we set $y = f(x)$ and solve for x in terms of y:

$$y = \frac{5-x}{3x+2}$$

- Multiply both sides by $(3x+2)$:

$$y(3x+2) = 5 - x$$

$$3xy + 2y = 5 - x$$

- Group all terms containing x on one side:

$$3xy + x = 5 - 2y$$

- Factor out x:

$$x(3y+1) = 5 - 2y$$

- Isolate x:

$$x = \frac{5-2y}{3y+1}$$

By swapping the variables x and y, we obtain the expression for the inverse function:
$\boxed{f^{-1}(x) = \frac{5-2x}{3x+1}, \quad x \neq -\frac{1}{3}}$

3. Solving the equation $f(\frac{1}{a}) = g(a+3)$ First, we express both sides of the equation in terms of a:

- Left-hand side (LHS):

$$f\left(\frac{1}{a}\right) = \frac{5 - \frac{1}{a}}{3\left(\frac{1}{a}\right) + 2} = \frac{\frac{5a-1}{a}}{\frac{3+2a}{a}} = \frac{5a-1}{2a+3}$$

- Right-hand side (RHS):

$$g(a+3) = 2(a+3) - 7 = 2a + 6 - 7 = 2a - 1$$

Equating the two expressions:

$$\frac{5a-1}{2a+3} = 2a - 1$$

- Multiply both sides by $(2a+3)$:

$$5a - 1 = (2a-1)(2a+3)$$

$$5a - 1 = 4a^2 + 6a - 2a - 3$$

$$5a - 1 = 4a^2 + 4a - 3$$

- Rearrange into a **quadratic equation** in standard form:

$$4a^2 - a - 2 = 0$$

- Apply the **quadratic formula** $a = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

$$a = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)}$$

$$= \frac{1 \pm \sqrt{1+32}}{8}$$

$$= \frac{1 \pm \sqrt{33}}{8}$$

$$\boxed{a = \frac{1 \pm \sqrt{33}}{8}}$$

WMA11_P3(IAL)_Summer_2022_Q3

Solution

1. Indefinite Integral Evaluation

To find the indefinite integral $\int \frac{9x}{3x^2+k}\,dx$, we employ the method of **u-substitution**. Let u be the expression in the denominator:

$$u = 3x^2 + k$$

Differentiating u with respect to x gives:

$$\frac{du}{dx} = 6x \implies dx = \frac{du}{6x}$$

Substituting these into the integral:

$$\begin{aligned}\int \frac{9x}{3x^2+k}\,dx &= \int \frac{9x}{u}\cdot\frac{du}{6x} \\ &= \int \frac{9}{6u}\,du \\ &= \frac{3}{2}\int \frac{1}{u}\,du \\ &= \frac{3}{2}\ln|\ u\ | + C\end{aligned}$$

Since k is a positive constant and $x^2 \geq 0$, the term $3x^2 + k$ is always positive. Thus, we can drop the absolute value signs. Substituting back for u:

$$\boxed{\frac{3}{2}\ln(3x^2+k) + C}$$

2. Finding the Value of k

We are given the definite integral:

$$\int_2^5 \frac{9x}{3x^2+k}\,dx = \ln 8$$

Using the **Fundamental Theorem of Calculus** and the result from part (a):

$$\left[\frac{3}{2}\ln(3x^2+k)\right]_2^5 = \ln 8$$

$$\frac{3}{2}(\ln(3(5)^2+k) - \ln(3(2)^2+k)) = \ln 8$$

$$\frac{3}{2}(\ln(75+k) - \ln(12+k)) = \ln 8$$

Multiply both sides by $\frac{2}{3}$ and use the **logarithm quotient rule**:

$$\ln\left(\frac{75+k}{12+k}\right) = \frac{2}{3}\ln 8$$

$$\ln\left(\frac{75+k}{12+k}\right) = \ln(8^{2/3})$$

$$\ln\left(\frac{75+k}{12+k}\right) = \ln((\sqrt[3]{8})^2)$$

$$\ln\left(\frac{75+k}{12+k}\right) = \ln 4$$

Since the natural logarithm is a one-to-one function, we equate the arguments:

$$\frac{75+k}{12+k} = 4$$

$$75+k = 4(12+k)$$

$$75+k = 48+4k$$

$$75-48 = 4k-k$$

$$27 = 3k$$

$$k = 9$$

The value of k is consistent with the given condition that k is a positive constant.

$$\boxed{k = 9}$$

WMA11_P3(IAL)_Summer_2022_Q4

Solution

The problem involves modeling the number of subscribers N to a streaming service using an exponential growth model $N = ab^t$, where t is the time in years. The relationship is linearized by taking the base-10 logarithm of both sides.

1. Find an equation for the line in Figure 1

The graph shows a linear relationship between t and $\log_{10} N$. Let $y = \log_{10} N$. The line passes through the points $(0, 3.08)$ and $(5, 3.85)$.

- The **gradient** m of the line is:

$$\begin{aligned} m &= \frac{y_2 - y_1}{t_2 - t_1} \\ &= \frac{3.85 - 3.08}{5 - 0} \\ &= \frac{0.77}{5} \\ &= 0.154 \end{aligned}$$

- The **y-intercept** c is given by the point $(0, 3.08)$, so $c = 3.08$.

The equation of the line is:

$$\boxed{\log_{10} N = 0.154t + 3.08}$$

2. Find the values of a and b

Starting from the model $N = ab^t$, we apply **logarithms** to both sides:

$$\begin{aligned} \log_{10} N &= \log_{10}(ab^t) \\ &= \log_{10} a + \log_{10}(b^t) \\ &= \log_{10} a + t \log_{10} b \end{aligned}$$

Comparing this to the linear equation $\log_{10} N = 3.08 + 0.154t$:

- $\log_{10} a = 3.08$
- $\log_{10} b = 0.154$

Solving for a:

$$\begin{aligned} a &= 10^{3.08} \\ &\approx 1202.26 \end{aligned}$$

To 3 significant figures, $a = 1200$.

Solving for b:

$$\begin{aligned} b &= 10^{0.154} \\ &\approx 1.4256 \end{aligned}$$

To 3 significant figures, $b = 1.43$.

$$\boxed{a = 1200, \quad b = 1.43}$$

3. Find the value of T when $N = 500\,000$

Substitute $N = 500\,000$ into the linear equation found in part (a):

$$\begin{aligned}
\log_{10}(500\,000) &= 0.154T + 3.08 \\
5.69897... &= 0.154T + 3.08 \\
0.154T &= 5.69897... - 3.08 \\
0.154T &= 2.61897... \\
T &= \frac{2.61897...}{0.154} \\
T &\approx 17.006
\end{aligned}$$

Rounding to an appropriate level of precision (usually 3 significant figures or the nearest integer for years):

$$\boxed{T = 17.0}$$

WMA11_P3(IAL)_Summer_2022_Q5

Solution

Based on the provided images, we are given the function $f(x) = |\ kx - 9\ | - 2$, where k is a positive constant. The graph of $y = f(x)$ is a V-shaped curve with a **minimum point** at B and a **y-intercept** at A.

1. Analysis of Point A and Point B

- **(i) Find the y-coordinate of A** The point A is the y-intercept, which occurs when $x = 0$.

$$\begin{aligned} y_A &= f(0) \\ &= |\ k(0) - 9\ | - 2 \\ &= |-9| - 2 \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

  The y-coordinate of A is 7.

- **(ii) Find, in terms of k, the x-coordinate of B** The minimum point B of an **absolute value function** of the form $y = |\ ax + b\ | + c$ occurs when the expression inside the absolute value is zero.

$$\begin{aligned} kx - 9 &= 0 \\ kx &= 9 \\ x_B &= \frac{9}{k} \end{aligned}$$

  The x-coordinate of B is $\frac{9}{k}$.

2. Solving the Inequality

We need to find the range of values of x that satisfy $|\ kx - 9\ | - 2 < 0$.

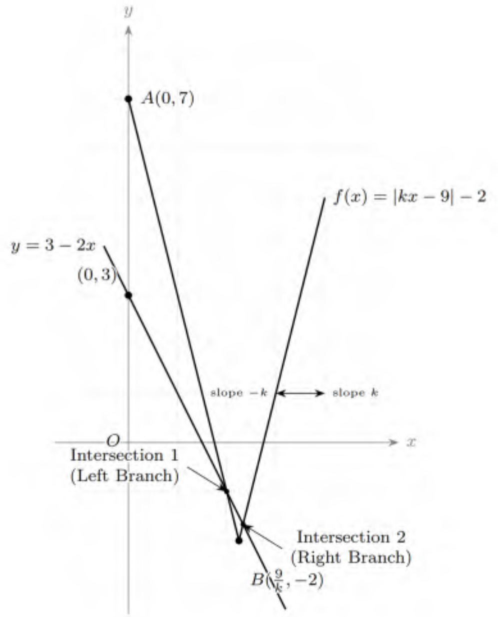
$$\begin{aligned} |\ kx - 9\ | &< 2 \\ -2 < kx - 9 &< 2 \\ 7 < kx &< 11 \end{aligned}$$

Since k is a positive constant, we divide the entire inequality by k:

$$\frac{7}{k} < x < \frac{11}{k}$$

3. Intersection with the line $y = 3 - 2x$

The line $y = 3 - 2x$ has a y-intercept of 3 and a slope of -2. For this line to intersect the graph $y = f(x)$ at two distinct points, we must consider the geometry of the V-shape.



The function $f(x)$ is defined piecewise:

$$f(x) = \begin{cases} -(kx-9)-2 = -kx+7 & \text{for } x \le \frac{9}{k} \\ (kx-9)-2 = kx-11 & \text{for } x > \frac{9}{k} \end{cases}$$

The line $y = 3 - 2x$ starts at $(0, 3)$, which is below point $A(0, 7)$. For the line to intersect the graph twice, it must intersect the left branch ($x < 9/k$) and the right branch ($x > 9/k$).

- **Left branch intersection:** The line $y = 3 - 2x$ intersects $y = -kx + 7$ if their slopes are not equal. However, since $3 < 7$, the line $y = 3 - 2x$ is already "inside" the V-shape at $x = 0$.
- **Condition for two points:** For the line to hit the right branch, it must pass "above" the vertex $B(9/k, -2)$. Since the line is always decreasing and the vertex is at $y = -2$, we check the slope. If the slope of the line (-2) is greater than the slope of the left branch ($-k$), the line will eventually "catch up" to the left branch.
- More critically, for two distinct intersections, the slope of the line must be between the slopes of the two branches, or the line must be steeper than the left branch.
- Specifically, the line $y = 3 - 2x$ will intersect the left branch if $-2 \neq -k$. It will intersect the right branch if the value of the line at the vertex $x = 9/k$ is greater than the y-coordinate of the vertex (-2):

$$3 - 2\left(\frac{9}{k}\right) > -2$$
$$5 > \frac{18}{k}$$
$$5k > 18$$
$$k > \frac{18}{5}$$

Additionally, if $k \le 2$, the line $y = 3 - 2x$ is steeper than or parallel to the left branch $y = -kx + 7$. Since the line starts below the branch at $x = 0$ ($3 < 7$) and is steeper, it will never cross the left branch for $x > 0$. Thus, we must have $k > 2$ to ensure an intersection on the left. Since $\frac{18}{5} = 3.6$, the condition $k > 3.6$ automatically satisfies $k > 2$.

Final Answers:

(a) (i) $y = 7$

(ii) $x = \frac{9}{k}$

(b) $\frac{7}{k} < x < \frac{11}{k}$

(c) $k > \frac{18}{5}$

WMA11_P3(IAL)_Summer_2022_Q6

Solution

The function f is defined for $x \geq -9/4$ by the expression:

$$f(x) = 5(x^2 - 2)(4x + 9)^{1/2}$$

1. Differentiation of $f(x)$ To find the derivative $f'(x)$, we apply the **product rule**, which states that for $y = uv$, $y' = u'v + uv'$. Let $u = 5(x^2 - 2)$ and $v = (4x + 9)^{1/2}$.

- $u' = 10x$
- $v' = \frac{1}{2}(4x + 9)^{-1/2} \cdot 4 = 2(4x + 9)^{-1/2}$

Applying the rule:

$$\begin{aligned} f'(x) &= 10x(4x + 9)^{1/2} + 5(x^2 - 2) \cdot 2(4x + 9)^{-1/2} \\ &= 10x(4x + 9)^{1/2} + \frac{10(x^2 - 2)}{(4x + 9)^{1/2}} \end{aligned}$$

To combine these terms, we find a common denominator of $(4x + 9)^{1/2}$:

$$\begin{aligned} f'(x) &= \frac{10x(4x + 9) + 10(x^2 - 2)}{(4x + 9)^{1/2}} \\ &= \frac{40x^2 + 90x + 10x^2 - 20}{(4x + 9)^{1/2}} \\ &= \frac{50x^2 + 90x - 20}{(4x + 9)^{1/2}} \\ &= \frac{10(5x^2 + 9x - 2)}{(4x + 9)^{1/2}} \end{aligned}$$

Comparing this to the required form $f'(x) = \frac{k(5x^2 + 9x - 2)}{(4x + 9)^{1/2}}$, we find $\boxed{k = 10}$.

2. Values of x for which $f'(x) = 0$ The derivative is zero when the numerator of the expression is zero:

$$5x^2 + 9x - 2 = 0$$

Using the **quadratic formula** or factoring:

$$(5x - 1)(x + 2) = 0$$

$$x = \frac{1}{5} \quad \text{or} \quad x = -2$$

Both values satisfy the domain constraint $x \geq -9/4$. $\boxed{x = -2, 0.2}$

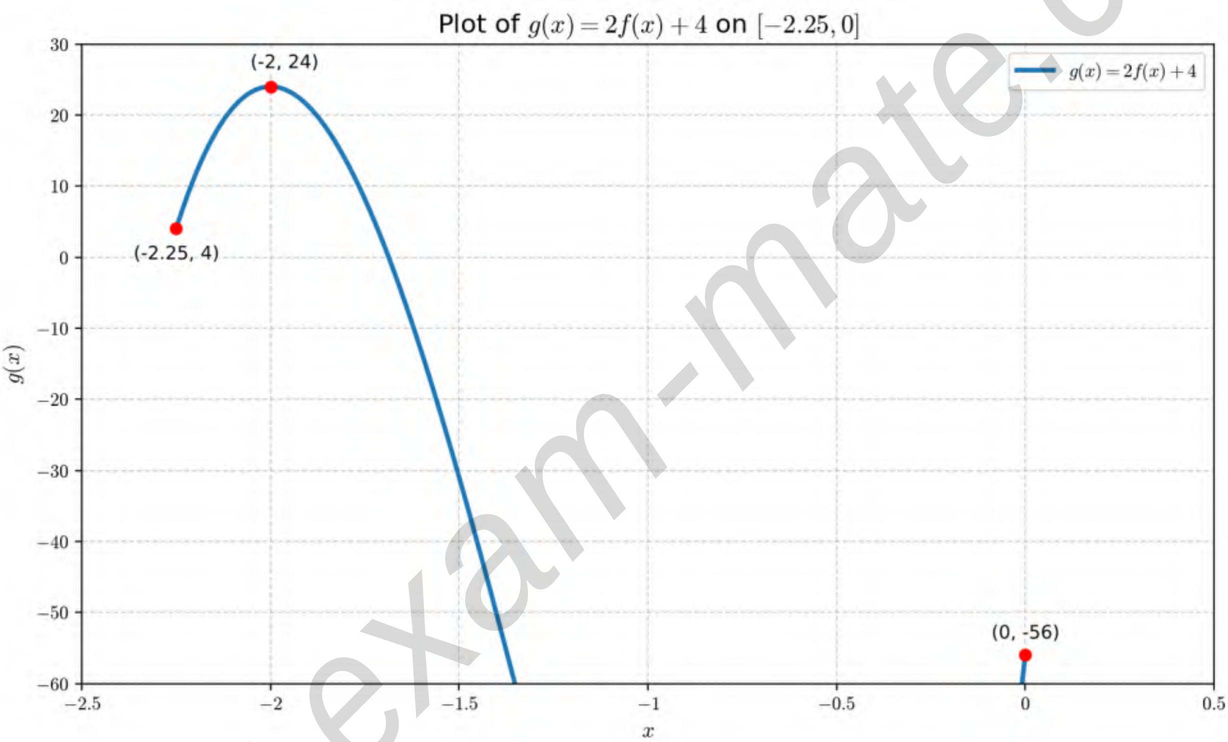
3. Exact coordinates of the local maximum P From Figure 3, the point P is a local maximum occurring at the smaller x-value in the domain. Comparing $x = -2$ and $x = 0.2$, the local maximum occurs at $x = -2$. To find the y-coordinate, substitute $x = -2$ into $f(x)$:

$$\begin{aligned} f(-2) &= 5((-2)^2 - 2)(4(-2)+9)^{1/2} \\ &= 5(4-2)(-8+9)^{1/2} \\ &= 5(2)(1)^{1/2} \\ &= 10 \end{aligned}$$

The coordinates of P are $\boxed{(-2, 10)}$.

4. Range of the function g The function g is defined as $g(x) = 2f(x) + 4$ for the restricted domain $-9/4 \leq x \leq 0$. To find the range, we examine the behavior of $f(x)$ on this interval:

- At the boundary $x = -9/4$: $f(-9/4) = 5((-9/4)^2 - 2)(0)^{1/2} = 0$.
- At the local maximum $x = -2$: $f(-2) = 10$.
- At the boundary $x = 0$: $f(0) = 5(0-2)(9)^{1/2} = 5(-2)(3) = -30$.



Now calculate the corresponding values for $g(x)$:

- $g(-9/4) = 2(0) + 4 = 4$
- $g(-2) = 2(10) + 4 = 24$
- $g(0) = 2(-30) + 4 = -56$

Since $f(x)$ is continuous, the range of $g(x)$ on $[-9/4, 0]$ is determined by the absolute minimum and maximum values on this interval. The maximum value is 24 and the minimum value is -56. $\boxed{-56 \leq g(x) \leq 24}$

WMA11_P3(IAL)_Summer_2022_Q7

Solution

1. Derivation of the Quadratic Equation in $\csc 2\theta$

To show that the equation $2\sin\theta(3\cot^2 2\theta - 7) = 13\sec\theta$ can be rewritten as $3\csc^2 2\theta - 13\csc 2\theta - 10 = 0$, we begin by expressing the **secant** and **cotangent** functions in terms of sine and cosine.

- First, rewrite the equation using $\sec\theta = \frac{1}{\cos\theta}$:

$$2\sin\theta(3\cot^2 2\theta - 7) = \frac{13}{\cos\theta}$$

- Multiply both sides by $\cos\theta$ to eliminate the fraction:

$$2\sin\theta\cos\theta(3\cot^2 2\theta - 7) = 13$$

- Apply the **double-angle identity** $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\sin 2\theta(3\cot^2 2\theta - 7) = 13$$

- Use the **Pythagorean identity** for cotangent, $\cot^2 2\theta = \csc^2 2\theta - 1$:

$$\sin 2\theta[3(\csc^2 2\theta - 1) - 7] = 13$$
$$\sin 2\theta(3\csc^2 2\theta - 3 - 7) = 13$$
$$\sin 2\theta(3\csc^2 2\theta - 10) = 13$$

- Distribute $\sin 2\theta$ across the terms. Since $\csc 2\theta = \frac{1}{\sin 2\theta}$, it follows that $\sin 2\theta \cdot \csc^2 2\theta = \csc 2\theta$:

$$3\csc 2\theta - 10\sin 2\theta = 13$$

- Multiply the entire equation by $\csc 2\theta$ (or divide by $\sin 2\theta$) to obtain a quadratic form in $\csc 2\theta$:

$$3\csc^2 2\theta - 10(\sin 2\theta\csc 2\theta) = 13\csc 2\theta$$
$$3\csc^2 2\theta - 10 = 13\csc 2\theta$$
$$3\csc^2 2\theta - 13\csc 2\theta - 10 = 0$$

2. Solving for θ in the interval $0 < \theta < \frac{\pi}{2}$

We solve the quadratic equation $3x^2 - 13x - 10 = 0$ where $x = \csc 2\theta$.

- Factor the quadratic:

$$(3x + 2)(x - 5) = 0$$
$$x = -\frac{2}{3} \quad \text{or} \quad x = 5$$

- Case 1: $\csc 2\theta = -\frac{2}{3}$ This implies $\sin 2\theta = -1.5$. Since the range of the **sine function** is $[-1, 1]$, this case yields no real solutions.
- Case 2: $\csc 2\theta = 5$ This implies $\sin 2\theta = \frac{1}{5} = 0.2$. We seek solutions for 2θ in the range $0 < 2\theta < \pi$ (since $0 < \theta < \frac{\pi}{2}$):

$$2\theta_1 = \arcsin(0.2) \approx 0.201358\,\text{rad}$$
$$2\theta_2 = \pi - \arcsin(0.2) \approx 2.940235\,\text{rad}$$

- Divide by 2 to find θ:

$$\theta_1 = \frac{0.201358}{2} \approx 0.100679\,\text{rad}$$
$$\theta_2 = \frac{2.940235}{2} \approx 1.470117\,\text{rad}$$

Rounding to 3 significant figures:

$$\boxed{\theta = 0.101, 1.47}$$

WMA11_P3(IAL)_Summer_2022_Q8

Solution

The velocity of a sprinter during a $100\,\mathrm{m}$ race is modeled by the function:

$$v(t) = 12 - e^{t-10} - 12e^{-0.75t}, \quad t \geq 0$$

1. Maximum Velocity To find the maximum velocity, we apply the **first derivative test** to identify the stationary points of $v(t)$.

- Differentiating $v(t)$ with respect to t:

$$\begin{aligned}\frac{dv}{dt} &= \frac{d}{dt}(12 - e^{t-10} - 12e^{-0.75t}) \\ &= -e^{t-10} - 12(-0.75)e^{-0.75t} \\ &= -e^{t-10} + 9e^{-0.75t}\end{aligned}$$

- Setting the derivative to zero for **extrema**:

$$\begin{aligned}-e^{t-10} + 9e^{-0.75t} &= 0 \\ e^{t-10} &= 9e^{-0.75t} \\ \frac{e^{t-10}}{e^{-0.75t}} &= 9 \\ e^{1.75t-10} &= 9\end{aligned}$$

- Taking the natural logarithm of both sides:

$$\begin{aligned}1.75t - 10 &= \ln(9) \\ 1.75t &= 10 + \ln(9) \\ t &= \frac{10 + \ln(9)}{1.75} \approx 6.970\,\mathrm{s}\end{aligned}$$

- Substituting this value back into the velocity equation:

$$\begin{aligned}v_{\max} &= 12 - e^{6.970-10} - 12e^{-0.75(6.970)} \\ &= 12 - e^{-3.030} - 12e^{-5.227} \\ &\approx 11.886\,\mathrm{m}\cdot\mathrm{s}^{-1}\end{aligned}$$

$$\boxed{11.886\,\mathrm{m}\cdot\mathrm{s}^{-1}}$$

2. Derivation of the Iteration Formula The total distance covered in T seconds is $100\,\mathrm{m}$, which is the integral of velocity:

$$\int_0^T (12 - e^{t-10} - 12e^{-0.75t})\,dt = 100$$

- Evaluating the **definite integral**:

$$\left[12t - e^{t-10} - \frac{12}{-0.75}e^{-0.75t}\right]_0^T = 100$$
$$\left[12t - e^{t-10} + 16e^{-0.75t}\right]_0^T = 100$$
$$(12T - e^{T-10} + 16e^{-0.75T}) - (12(0) - e^{-10} + 16e^{0}) = 100$$
$$12T - e^{T-10} + 16e^{-0.75T} + e^{-10} - 16 = 100$$

- Rearranging to solve for T:

$$12T = 116 + e^{T-10} - 16e^{-0.75T} - e^{-10}$$
$$T = \frac{1}{12}(116 - 16e^{-0.75T} + e^{T-10} - e^{-10})$$

This matches the required form for the **fixed-point iteration**.

3. Numerical Approximation of T Using the iteration formula $T_{n+1} = \frac{1}{12}(116 - 16e^{-0.75T_n} + e^{T_n-10} - e^{-10})$ with $T_1 = 10$:

- **(i) Value of T_2**

$$T_2 = \frac{1}{12}(116 - 16e^{-0.75(10)} + e^{10-10} - e^{-10})$$
$$= \frac{1}{12}(116 - 16e^{-7.5} + 1 - e^{-10})$$
$$\approx 9.7492$$

$\boxed{T_2 = 9.7492}$

- **(ii) Time taken for the race** Continuing the iteration until convergence to 4 decimal places:

$$T_3 = \frac{1}{12}(116 - 16e^{-0.75(9.7492)} + e^{9.7492-10} - e^{-10}) \approx 9.7329$$
$$T_4 = \frac{1}{12}(116 - 16e^{-0.75(9.7329)} + e^{9.7329-10} - e^{-10}) \approx 9.7319$$
$$T_5 = \frac{1}{12}(116 - 16e^{-0.75(9.7319)} + e^{9.7319-10} - e^{-10}) \approx 9.7318$$
$$T_6 = \frac{1}{12}(116 - 16e^{-0.75(9.7318)} + e^{9.7318-10} - e^{-10}) \approx 9.7318$$

The value stabilizes at $9.7318\,\text{s}$. $\boxed{9.7318\,\text{s}}$

WMA11_P3(IAL)_Summer_2022_Q9

Solution

The curve is defined by the equation:

$$y = \frac{1 + 2\cos x}{1 + \sin x}, \quad -\frac{\pi}{2} < x < \frac{3\pi}{2}$$

1. Finding the derivative to locate the minimum point M To find the x-coordinate of the **stationary point** M, we must calculate the derivative dy/dx and set it to zero. We apply the **quotient rule**:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let $u = 1 + 2\cos x$ and $v = 1 + \sin x$. Then:

- $\frac{du}{dx} = -2\sin x$
- $\frac{dv}{dx} = \cos x$

Substituting these into the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1 + \sin x)(-2\sin x) - (1 + 2\cos x)(\cos x)}{(1 + \sin x)^2} \\ &= \frac{-2\sin x - 2\sin^2 x - \cos x - 2\cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-2\sin x - \cos x - 2(\sin^2 x + \cos^2 x)}{(1 + \sin x)^2}\end{aligned}$$

Using the **Pythagorean identity** $\sin^2 x + \cos^2 x = 1$:

$$\frac{dy}{dx} = \frac{-2\sin x - \cos x - 2}{(1 + \sin x)^2}$$

At the minimum point M, the gradient is zero ($dy/dx = 0$):

$$\begin{aligned}\frac{-2\sin x - \cos x - 2}{(1 + \sin x)^2} &= 0 \\ -2\sin x - \cos x - 2 &= 0 \\ 2\sin x + \cos x &= -2\end{aligned}$$

Thus, the x-coordinate of M is a solution to $2\sin x + \cos x = -2$.

2. Solving for the x-coordinate of M To solve $2\sin x + \cos x = -2$, we use the **R-formula** method, expressing the left-hand side in the form $R\cos(x - \alpha)$.

- $R = \sqrt{2^2 + 1^2} = \sqrt{5}$
- $\tan\alpha = \frac{2}{1} \implies \alpha = \arctan(2) \approx 1.1071\,\text{rad}$

The equation becomes:

$$\sqrt{5}\cos(x - 1.1071) = -2$$
$$\cos(x - 1.1071) = -\frac{2}{\sqrt{5}}$$

Finding the **principal value**:

$$\arccos\left(-\frac{2}{\sqrt{5}}\right) \approx 2.6779\,\text{rad}$$

The general solutions for $x - 1.1071$ are:

$$x - 1.1071 = \pm 2.6779 + 2n\pi$$

For $n = 0$:

- $x_1 = 2.6779 + 1.1071 = 3.7850$
- $x_2 = -2.6779 + 1.1071 = -1.5708$ (which is $-\pi/2$)

From Figure 5, M is located in the region where $x > 0$. The value $x = -\pi/2$ corresponds to a vertical asymptote where the denominator $1 + \sin x = 0$. Therefore, the x-coordinate of M is 3.7850.

Rounding to 3 significant figures:

$$x \approx 3.79$$

$\boxed{x = 3.79}$

WMA11_P3(IAL)_Winter_2022_Q1

Solution

To find the x-coordinate of the **stationary point** for the curve defined by the equation $y = (2x+5)e^{3x}$, we must determine the value of x where the first derivative dy/dx is equal to zero.

1. Differentiate the function The function is a product of two terms: $u(x) = 2x + 5$ and $v(x) = e^{3x}$. We apply the **product rule**, which states:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

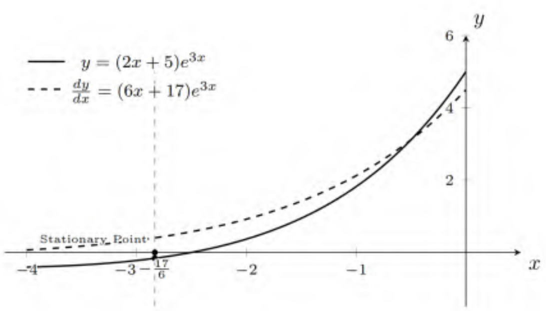
- Let $u = 2x + 5 \implies \frac{du}{dx} = 2$
- Let $v = e^{3x} \implies \frac{dv}{dx} = 3e^{3x}$ (using the **chain rule**)

Substituting these into the product rule formula:

$$\begin{aligned}\frac{dy}{dx} &= (2x+5)(3e^{3x}) + (e^{3x})(2) \\ &= 3e^{3x}(2x+5) + 2e^{3x}\end{aligned}$$

2. Simplify the derivative Factor out the common term e^{3x}:

$$\begin{aligned}\frac{dy}{dx} &= e^{3x}[3(2x+5)+2] \\ &= e^{3x}[6x+15+2] \\ &= e^{3x}(6x+17)\end{aligned}$$



3. Solve for the stationary point A stationary point occurs when $\frac{dy}{dx} = 0$:

$$e^{3x}(6x + 17) = 0$$

Since the **exponential function** e^{3x} is never zero for any real value of x, we must have:

$$6x + 17 = 0$$
$$6x = -17$$
$$x = -\frac{17}{6}$$

The x-coordinate of the stationary point is:

$$\boxed{x = -\frac{17}{6}}$$

WMA11_P3(IAL)_Winter_2022_Q2

Solution

1. Transformation of the Equation

To rewrite the equation $8\cos\theta = 3\csc\theta$ in the form $\sin 2\theta = k$, we use the fundamental **trigonometric identity** for the **cosecant** function:

$$\csc\theta = \frac{1}{\sin\theta}$$

- Substitute this into the original equation:

$$8\cos\theta = 3\left(\frac{1}{\sin\theta}\right)$$

- Multiply both sides by $\sin\theta$ (assuming $\sin\theta \neq 0$):

$$8\sin\theta\cos\theta = 3$$

- Recall the **double-angle formula** for sine:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

- We can rewrite $8\sin\theta\cos\theta$ as $4(2\sin\theta\cos\theta)$:

$$\begin{aligned} 4(2\sin\theta\cos\theta) &= 3 \\ 4\sin 2\theta &= 3 \\ \sin 2\theta &= \frac{3}{4} \end{aligned}$$

Comparing this to the form $\sin 2\theta = k$, we find that $k = 0.75$.

2. Finding the Smallest Positive Solution

We now solve the equation $\sin 2\theta = 0.75$ for the smallest positive value of θ.

- First, find the **principal value** for 2θ using the inverse sine function:

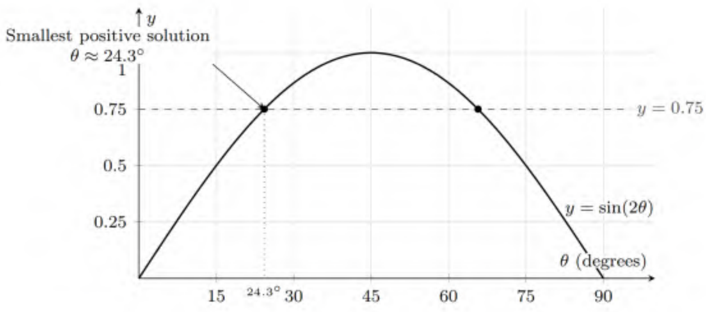
$$2\theta = \arcsin(0.75)$$

- Calculating the value in degrees:

$$\begin{aligned} 2\theta &\approx 48.59037789^\circ \\ \theta &\approx \frac{48.59037789^\circ}{2} \\ \theta &\approx 24.29518895^\circ \end{aligned}$$

- Rounding to one decimal place as requested:

$$\theta \approx 24.3^\circ$$


y
Smallest positive solution
$\theta \approx 24.3°$
1
0.75
0.5
0.25
$y = 0.75$
$y = \sin(2\theta)$
θ (degrees)
15
24.3°
30
45
60
75
90

$\boxed{24.3°}$

WMA11_P3(IAL)_Winter_2022_Q3

Solution

1. Evaluation of the Indefinite Integral

To find the indefinite integral $\int (2x-5)^7\,dx$, we apply the **power rule for integration** combined with the **linear substitution** method.

- Let $u = 2x - 5$. Then the derivative is $\frac{du}{dx} = 2$, which implies $dx = \frac{1}{2}du$.
- Substituting these into the integral:

$$\begin{aligned}\int (2x-5)^7\,dx &= \int u^7 \cdot \frac{1}{2}\,du \\ &= \frac{1}{2}\int u^7\,du \\ &= \frac{1}{2}\left(\frac{u^{7+1}}{7+1}\right) + C \\ &= \frac{1}{2}\left(\frac{u^8}{8}\right) + C \\ &= \frac{1}{16}(2x-5)^8 + C\end{aligned}$$

 where C is the **constant of integration**.

$$\boxed{\frac{1}{16}(2x-5)^8 + C}$$

2. Evaluation of the Definite Integral

We are required to show that $\int_0^{\frac{\pi}{3}} \frac{4\sin x}{1+2\cos x}\,dx = \ln a$ and find the rational constant a.

- We use the **u-substitution** method. Let $u = 1 + 2\cos x$.
- Differentiating u with respect to x:

$$\begin{aligned}\frac{du}{dx} &= -2\sin x \\ du &= -2\sin x\,dx \\ -2\,du &= 4\sin x\,dx\end{aligned}$$

- Next, we determine the new **limits of integration** for u:
  - Lower limit: When $x = 0$, $u = 1 + 2\cos(0) = 1 + 2(1) = 3$.
  - Upper limit: When $x = \frac{\pi}{3}$, $u = 1 + 2\cos\left(\frac{\pi}{3}\right) = 1 + 2\left(\frac{1}{2}\right) = 2$.
- Substituting these into the definite integral:

$$\int_0^{\frac{\pi}{3}} \frac{4\sin x}{1+2\cos x}\,dx = \int_3^2 \frac{-2}{u}\,du$$
$$= -2\int_3^2 \frac{1}{u}\,du$$
$$= -2[\ln|\ u\ |]_3^2$$
$$= -2(\ln 2 - \ln 3)$$
$$= 2(\ln 3 - \ln 2)$$

- Using the **logarithm laws** ($\ln x - \ln y = \ln \frac{x}{y}$ and $n\ln x = \ln x^n$):

$$2(\ln 3 - \ln 2) = 2\ln\left(\frac{3}{2}\right)$$
$$= \ln\left(\frac{3}{2}\right)^2$$
$$= \ln\left(\frac{9}{4}\right)$$

Comparing this to the form $\ln a$, we find $a = \frac{9}{4}$. Since 9 and 4 are integers, a is a rational constant.

$$\boxed{a = \frac{9}{4}}$$

WMA11_P3(IAL)_Winter_2022_Q4

Solution

The growth of the weed on the pond surface is modeled by the **logistic growth** equation:

$$A = \frac{80pe^{0.15t}}{pe^{0.15t} + 4}$$

where A is the area in m^2, t is the time in days, and p is a positive constant.

1. Show that $p = 2.4$ - At the start of the study ($t = 0$), the area covered is $A = 30\,\mathrm{m}^2$. - Substituting these values into the model:

$$30 = \frac{80pe^{0.15(0)}}{pe^{0.15(0)} + 4}$$
$$30 = \frac{80p(1)}{p(1) + 4}$$
$$30(p + 4) = 80p$$
$$30p + 120 = 80p$$
$$50p = 120$$
$$p = \frac{120}{50} = 2.4$$

- Thus, it is shown that $p = 2.4$.

2. Find the value of T - We are given that $A = 50\,\mathrm{m}^2$ when $t = T$. Using $p = 2.4$:

$$50 = \frac{80(2.4)e^{0.15T}}{2.4e^{0.15T} + 4}$$
$$50(2.4e^{0.15T} + 4) = 192e^{0.15T}$$
$$120e^{0.15T} + 200 = 192e^{0.15T}$$
$$200 = 192e^{0.15T} - 120e^{0.15T}$$
$$200 = 72e^{0.15T}$$
$$e^{0.15T} = \frac{200}{72} = \frac{25}{9}$$

- To solve for T, we take the **natural logarithm** of both sides:

$$0.15T = \ln\left(\frac{25}{9}\right)$$
$$T = \frac{\ln(25/9)}{0.15}$$
$$T \approx 6.811$$

- Rounding to one decimal place, we obtain $T = 6.8$.

3. Maximum possible surface area of the pond - The maximum area occurs as $t \to \infty$. This represents the **carrying capacity** of the model. - We analyze the limit of the expression by dividing the numerator and denominator by $e^{0.15t}$:

$$A = \frac{80p}{p + 4e^{-0.15t}}$$

$$\lim_{t \to \infty} A = \frac{80p}{p + 4(0)} = 80$$

- Alternatively, as t becomes very large, the term 4 in the denominator becomes negligible compared to $pe^{0.15t}$:

$$A \approx \frac{80pe^{0.15t}}{pe^{0.15t}} = 80$$

[Visualization]

(a) $p = 2.4$ (as shown)
(b) $\boxed{T = 6.8}$
(c) $\boxed{80\,\mathrm{m}^2}$

WMA11_P3(IAL)_Winter_2022_Q5

Solution

The curve is defined by the equation:

$$f(x) = 6\ln(2x+3) - \frac{1}{2}x^2 + 4, \quad x > -1.5$$

1. Interval for the x-coordinate of P To show that a root lies in the interval $[-1.25, -1.2]$, we apply the **Intermediate Value Theorem** by checking for a sign change in $f(x)$ at the boundaries.

- At $x = -1.25$:

$$\begin{aligned} f(-1.25) &= 6\ln(2(-1.25)+3) - \frac{1}{2}(-1.25)^2 + 4 \\ &= 6\ln(0.5) - 0.78125 + 4 \\ &\approx -4.15888 - 0.78125 + 4 \\ &\approx -0.9401 \end{aligned}$$

- At $x = -1.2$:

$$\begin{aligned} f(-1.2) &= 6\ln(2(-1.2)+3) - \frac{1}{2}(-1.2)^2 + 4 \\ &= 6\ln(0.6) - 0.72 + 4 \\ &\approx -3.06495 - 0.72 + 4 \\ &\approx 0.2151 \end{aligned}$$

Since $f(-1.25) < 0$ and $f(-1.2) > 0$, and the function is continuous on this interval, there must be at least one root $x \in [-1.25, -1.2]$.

2. Iterative process for the x-coordinate of Q The iterative formula is given as $x_{n+1} = \sqrt{12\ln(2x_n+3)+8}$ with $x_1 = 6$.

- (i) Finding x_2:

$$\begin{aligned} x_2 &= \sqrt{12\ln(2(6)+3)+8} \\ &= \sqrt{12\ln(15)+8} \\ &\approx \sqrt{12(2.70805)+8} \\ &\approx \sqrt{40.4966} \\ &\approx 6.36369... \end{aligned}$$

To 4 decimal places, $x_2 = 6.3637$.

- (ii) Continued iteration for Q: Using the **Fixed-point iteration** method:

$$x_3 = \sqrt{12\ln(2(6.36369)+3)+8} \approx 6.4377$$
$$x_4 = \sqrt{12\ln(2(6.4377)+3)+8} \approx 6.4523$$
$$x_5 = \sqrt{12\ln(2(6.4523)+3)+8} \approx 6.4552$$
$$x_6 = \sqrt{12\ln(2(6.4552)+3)+8} \approx 6.4558$$
$$x_7 = \sqrt{12\ln(2(6.4558)+3)+8} \approx 6.4559$$
$$x_8 = \sqrt{12\ln(2(6.4559)+3)+8} \approx 6.4559$$

The x-coordinate of Q is 6.4559.

3. Finding the x-coordinate of the maximum M To find the turning point M, we calculate the first derivative $f'(x)$ and set it to zero.

- Differentiating $f(x)$:

$$\begin{aligned}\frac{dy}{dx} &= 6\cdot\frac{d}{dx}(\ln(2x+3)) - \frac{1}{2}\cdot\frac{d}{dx}(x^2) + \frac{d}{dx}(4)\\ &= 6\cdot\frac{2}{2x+3} - x\\ &= \frac{12}{2x+3} - x\end{aligned}$$

- Setting $\frac{dy}{dx} = 0$:

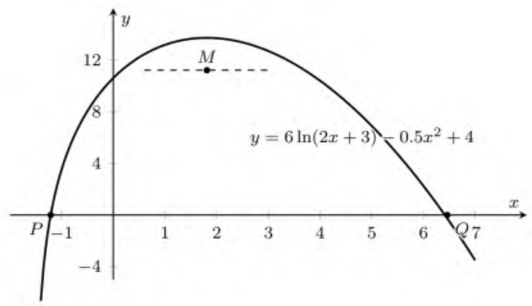
$$\begin{aligned}\frac{12}{2x+3} - x &= 0\\ 12 &= x(2x+3)\\ 12 &= 2x^2+3x\\ 2x^2+3x-12 &= 0\end{aligned}$$

- Solving the **Quadratic Equation** using the quadratic formula:

$$\begin{aligned}x &= \frac{-3 \pm \sqrt{3^2-4(2)(-12)}}{2(2)}\\ &= \frac{-3 \pm \sqrt{9+96}}{4}\\ &= \frac{-3 \pm \sqrt{105}}{4}\end{aligned}$$

Since M is in the first quadrant (as seen in Figure 1) and $x > -1.5$, we take the positive root:

$$\begin{aligned}x &= \frac{-3+\sqrt{105}}{4}\\ &\approx \frac{-3+10.24695}{4}\\ &\approx 1.8117\end{aligned}$$



(a) $f(-1.25) \approx -0.9401$, $f(-1.2) \approx 0.2151$. Sign change implies root in $[-1.25, -1.2]$.

(b) (i) $\boxed{6.3637}$ (ii) $\boxed{6.4559}$

(c) $\boxed{\dfrac{-3+\sqrt{105}}{4}}$ or $\boxed{1.8117}$

WMA11_P3(IAL)_Winter_2022_Q6

Solution

The function f is defined by:

$$f(x) = \frac{5x-3}{x-4}, \quad x > 4$$

1. Show that f is a decreasing function To determine the monotonicity of the function, we calculate its first derivative using the **quotient rule**:

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(5x-3)\cdot(x-4) - (5x-3)\cdot\frac{d}{dx}(x-4)}{(x-4)^2} \\ &= \frac{5(x-4)-(5x-3)(1)}{(x-4)^2} \\ &= \frac{5x-20-5x+3}{(x-4)^2} \\ &= \frac{-17}{(x-4)^2} \end{aligned}$$

For all x in the domain $x > 4$, the denominator $(x-4)^2$ is strictly positive. Since the numerator is -17, it follows that $f'(x) < 0$ for all $x > 4$. Therefore, f is a **decreasing function**.

2. Find the inverse function f^{-1} To find the **inverse function**, we set $y = f(x)$ and solve for x:

$$\begin{aligned} y &= \frac{5x-3}{x-4} \\ y(x-4) &= 5x-3 \\ yx-4y &= 5x-3 \\ yx-5x &= 4y-3 \\ x(y-5) &= 4y-3 \\ x &= \frac{4y-3}{y-5} \end{aligned}$$

Interchanging x and y, we obtain:

$$\boxed{f^{-1}(x) = \frac{4x-3}{x-5}}$$

The domain of f^{-1} is the range of f. As $x \to 4^+$, $f(x) \to \infty$, and as $x \to \infty$, $f(x) \to 5$. Thus, the domain of f^{-1} is $x > 5$.

3. Analyze the composite function $ff(x)$ (i) We evaluate the **composite function** $f(f(x))$:

$$ff(x) = \frac{5\left(\frac{5x-3}{x-4}\right) - 3}{\left(\frac{5x-3}{x-4}\right) - 4}$$
$$= \frac{\frac{5(5x-3)-3(x-4)}{x-4}}{\frac{(5x-3)-4(x-4)}{x-4}}$$
$$= \frac{25x - 15 - 3x + 12}{5x - 3 - 4x + 16}$$
$$= \frac{22x - 3}{x + 13}$$

Comparing this to the form $\frac{ax+b}{x+c}$, we find the constants:

$$\boxed{a = 22, \quad b = -3, \quad c = 13}$$

(ii) To deduce the range of ff, we consider the domain of f, which is $x > 4$. The range of f for $x > 4$ is $(5, \infty)$. The composite function $ff(x)$ is defined for x such that x is in the domain of f ($x > 4$) and $f(x)$ is in the domain of f ($f(x) > 4$). Since the range of f is $(5, \infty)$, and $5 > 4$, all values of $f(x)$ are valid inputs for the second application of f. We examine the behavior of $ff(x) = \frac{22x-3}{x+13}$ on the interval $x \in (4, \infty)$:

- At the boundary $x = 4$: $ff(4) = \frac{22(4)-3}{4+13} = \frac{88-3}{17} = \frac{85}{17} = 5$.
- As $x \to \infty$: $\lim_{x\to\infty} \frac{22x-3}{x+13} = 22$.

Since $ff(x)$ is a **linear fractional transformation** and its derivative $ff'(x) = \frac{22(13)-(-3)(1)}{(x+13)^2} = \frac{289}{(x+13)^2} > 0$, the function is strictly increasing. Thus, the range is $(5, 22)$.

$$\boxed{5 < ff(x) < 22}$$

WMA11_P3(IAL)_Winter_2022_Q7

Solution

The function is defined as $f(x) = \frac{1}{2} \mid 2x + 7 \mid - 10$.

1. Coordinates of the vertex V The **vertex** of an **absolute value function** of the form $y = a \mid bx + c \mid + d$ occurs where the expression inside the absolute value is zero.

- Set $2x + 7 = 0$:

$$2x = -7$$
$$x = -3.5$$

- Substitute $x = -3.5$ into $f(x)$:

$$f(-3.5) = \frac{1}{2} \mid 0 \mid - 10$$
$$= -10$$

The coordinates of the vertex V are $\boxed{(-3.5, -10)}$.

2. Algebraic solution of the inequality We solve $\frac{1}{2} \mid 2x + 7 \mid - 10 \geq \frac{1}{3}x + 1$. First, isolate the absolute value term:

$$\frac{1}{2} \mid 2x + 7 \mid \geq \frac{1}{3}x + 11$$
$$\mid 2x + 7 \mid \geq \frac{2}{3}x + 22$$

This inequality is satisfied if $2x + 7 \geq \frac{2}{3}x + 22$ or $2x + 7 \leq -(\frac{2}{3}x + 22)$.

- **Case 1:** $2x + 7 \geq \frac{2}{3}x + 22$

$$2x - \frac{2}{3}x \geq 22 - 7$$
$$\frac{4}{3}x \geq 15$$
$$x \geq 11.25$$

- **Case 2:** $2x + 7 \leq -\frac{2}{3}x - 22$

$$2x + \frac{2}{3}x \leq -22 - 7$$
$$\frac{8}{3}x \leq -29$$
$$x \leq -10.875$$

The solution set is $\boxed{x \leq -10.875 \quad \text{or} \quad x \geq 11.25}$.

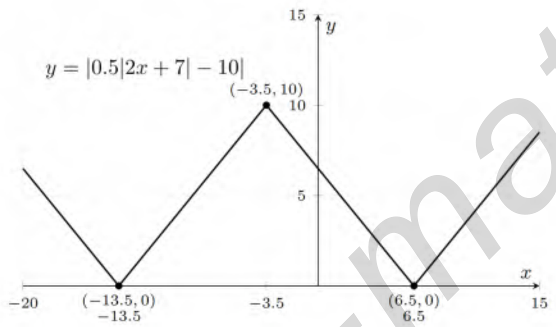
3. Sketch of $y = \mid f(x) \mid$ The graph of $y = \mid f(x) \mid$ is obtained by reflecting the portions of $y = f(x)$ that lie below the x-axis across the x-axis.

- **Local Minimum Points:** These occur at the original x-intercepts of $f(x)$, where $\mid f(x) \mid = 0$.

$$\frac{1}{2} \mid 2x+7 \mid -10 = 0$$
$$\mid 2x+7 \mid = 20$$

Solving $2x+7=20$ gives $x=6.5$. Solving $2x+7=-20$ gives $x=-13.5$. The local minima are at $(-13.5, 0)$ and $(6.5, 0)$.

- **Local Maximum Point:** The original vertex $V(-3.5, -10)$ is reflected to $(-3.5, 10)$. Since the original graph was decreasing then increasing, the reflected vertex becomes a local maximum.



- Local maximum point: $\boxed{(-3.5, 10)}$
- Local minimum points: $\boxed{(-13.5, 0) \quad \text{and} \quad (6.5, 0)}$

WMA11_P3(IAL)_Winter_2022_Q8

Solution

1. Conversion to Exponential Form

The given equation for the amount of antibiotic x (in milligrams) at time t (in hours) is:

$$\log_{10} x = 2.74 - 0.079t$$

To express this in the form $x = pq^{-t}$, we apply the definition of a **logarithm** (base 10) to both sides:

$$\begin{aligned} x &= 10^{2.74-0.079t} \\ &= 10^{2.74} \cdot 10^{-0.079t} \\ &= 10^{2.74} \cdot (10^{0.079})^{-t} \end{aligned}$$

Comparing this to the form $x = pq^{-t}$, we identify:

- $p = 10^{2.74}$
- $q = 10^{0.079}$

Calculating the numerical values:

- $p = 10^{2.74} \approx 549.54$
- $q = 10^{0.079} \approx 1.1995$

Rounding p to the nearest whole number and q to 2 significant figures:

- $p = 550$
- $q = 1.2$

The equation is $x = 550(1.2)^{-t}$.

2. Interpretation of Constant p

In the equation $x = pq^{-t}$, when $t = 0$:

$$x = pq^0 = p$$

Thus, the constant p represents the **initial dose** (or initial amount) of the antibiotic in the patient's bloodstream at the moment the dose is given ($t = 0$).

3. Rate of Change via Calculus

For a different patient, the equation is given as:

$$x = 400 \times 1.4^{-t}$$

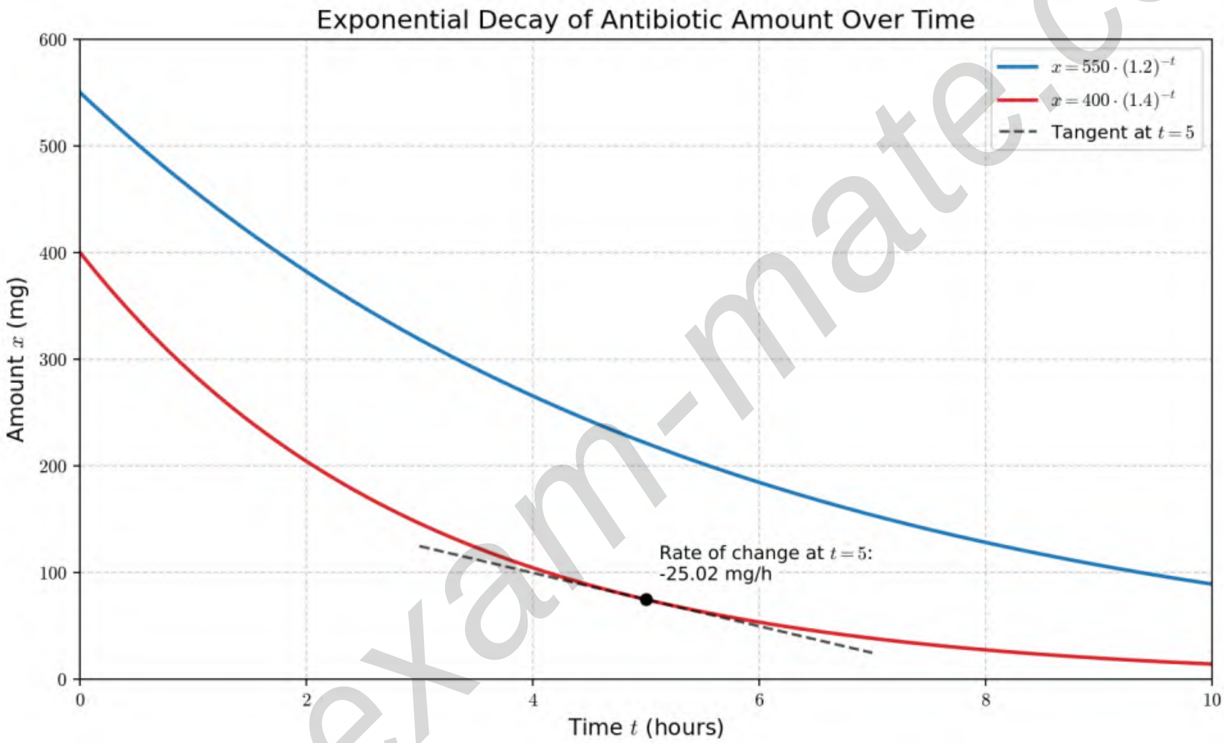
To find the **rate of change** dx/dt, we use the rule for differentiating **exponential functions** of the form a^u:

$$\frac{dx}{dt} = \frac{d}{dt}(400 \cdot 1.4^{-t})$$
$$= 400 \cdot \ln(1.4) \cdot 1.4^{-t} \cdot \frac{d}{dt}(-t)$$
$$= -400\ln(1.4) \cdot 1.4^{-t}$$

Evaluating the derivative at $t = 5$:

$$\frac{dx}{dt}|_{t=5} = -400\ln(1.4) \cdot 1.4^{-5}$$
$$\approx -400 \cdot 0.33647 \cdot 0.18593$$
$$\approx -25.023$$

Rounding to 2 significant figures, the value is -25.



(a) $p = 550$, $q = 1.2$
(b) p is the initial amount of antibiotic in the bloodstream (at $t = 0$).
(c) $\boxed{-25}$

WMA11_P3(IAL)_Winter_2022_Q9

Solution

1. Solving the Trigonometric Equation

To solve the equation $2\sec^2 x - 3\tan x = 2$ for $0 < x \leq \pi$, we first express all terms in terms of $\tan x$ using the **Pythagorean identity**:

$$\sec^2 x = 1 + \tan^2 x$$

- **Substitution and Simplification:** Substituting this into the original equation:

$$2(1 + \tan^2 x) - 3\tan x = 2$$
$$2 + 2\tan^2 x - 3\tan x = 2$$
$$2\tan^2 x - 3\tan x = 0$$

- **Factoring the Quadratic:** Factor out the common term $\tan x$:

$$\tan x(2\tan x - 3) = 0$$

  This yields two possible cases:

  1. $\tan x = 0$
  2. $2\tan x - 3 = 0 \implies \tan x = 1.5$

- **Finding Solutions in the Interval $0 < x \leq \pi$:**
  - For $\tan x = 0$: In the interval $(0, \pi]$, the only solution is $x = \pi$.
  - For $\tan x = 1.5$: Using the inverse tangent function:

$$x = \arctan(1.5) \approx 0.98279...$$

    Since 1.5 is positive, the principal value lies in the first quadrant, which is within our range.

- **Final Values (to 3 significant figures):**
  - $x = 0.983\,\text{rad}$
  - $x = 3.14\,\text{rad}$ (which is π)

$\boxed{x = 0.983, 3.14}$

2. Proving the Trigonometric Identity

To prove that $\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$, we start by combining the fractions over a common denominator.

- **Combining Fractions:**

$$\text{LHS} = \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

- **Applying Compound Angle Identities:** The numerator matches the form of the **sine subtraction formula**: $\sin(A - B) = \sin A\cos B - \cos A\sin B$. Let $A = 3\theta$ and $B = \theta$:

$$\sin 3\theta \cos\theta - \cos 3\theta \sin\theta = \sin(3\theta - \theta) = \sin 2\theta$$

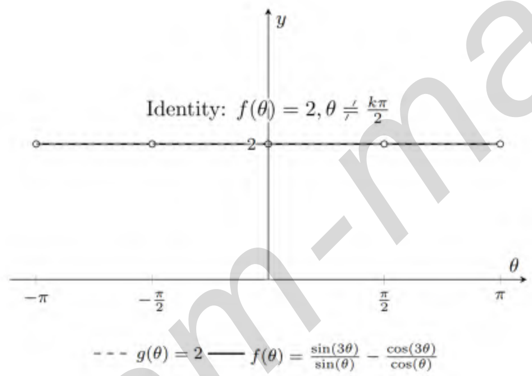
Thus, the expression becomes:

$$\text{LHS} = \frac{\sin 2\theta}{\sin\theta\cos\theta}$$

- **Applying Double Angle Identities:** Recall the **double angle formula** for sine: $\sin 2\theta = 2\sin\theta\cos\theta$. Substituting this into the numerator:

$$\text{LHS} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$= 2$$

The left-hand side equals the right-hand side, completing the proof.



WMA11_P3(IAL)_Winter_2022_Q10

Solution

The curve C is defined by the equation:

$$x = ye^{2y}, \quad y \in \mathbb{R}$$

1. Differentiation of the curve

To find the derivative $\frac{dy}{dx}$, we apply **implicit differentiation** with respect to x (or differentiate x with respect to y and take the reciprocal). Differentiating x with respect to y using the **product rule**:

$$\begin{aligned}\frac{dx}{dy} &= \frac{d}{dy}(y) \cdot e^{2y} + y \cdot \frac{d}{dy}(e^{2y}) \\ &= 1 \cdot e^{2y} + y \cdot (2e^{2y}) \\ &= e^{2y}(1+2y)\end{aligned}$$

Since $\frac{dy}{dx} = \frac{1}{dx/dy}$, we have:

$$\frac{dy}{dx} = \frac{1}{e^{2y}(1+2y)}$$

From the original equation, we know $e^{2y} = \frac{x}{y}$. Substituting this into the expression:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\left(\frac{x}{y}\right)(1+2y)} \\ &= \frac{y}{x(1+2y)}\end{aligned}$$

This confirms the required result.

2. Finding the range of k for two intersections

The line $x = k$ intersects the curve C at exactly two points if the equation $k = ye^{2y}$ has exactly two distinct real solutions for y. This is a problem of finding the range of the function $f(y) = ye^{2y}$ where it is not **monotonic**.

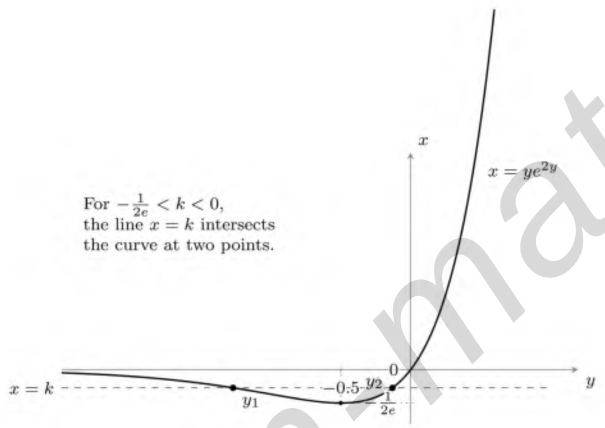
- **Stationary points:** We find where the derivative of x with respect to y is zero:

$$\begin{aligned}\frac{dx}{dy} = e^{2y}(1+2y) &= 0 \\ 1 + 2y &= 0 \\ y &= -\frac{1}{2}\end{aligned}$$

- **Corresponding x-value:** At $y = -\frac{1}{2}$:

$$x = \left(-\frac{1}{2}\right)e^{2(-1/2)}$$
$$= -\frac{1}{2}e^{-1} = -\frac{1}{2e}$$

- **Asymptotic behavior:**
  - As $y \to \infty$, $x = ye^{2y} \to \infty$.
  - As $y \to -\infty$, $x = ye^{2y} \to 0$ (since the exponential term decays faster than the linear term grows).



By analyzing the graph or the function's behavior:

- If $k < -\frac{1}{2e}$, there are no solutions.
- If $k = -\frac{1}{2e}$, there is exactly one solution (the local minimum).
- If $-\frac{1}{2e} < k < 0$, the line $x = k$ intersects the "loop" part of the curve twice (once for $y < -1/2$ and once for $-1/2 < y < 0$).
- If $k \geq 0$, the line intersects the curve only once (at $y \geq 0$).

Therefore, for exactly two points of intersection, k must be in the interval:

$$\boxed{-\frac{1}{2e} < k < 0}$$

WMA11_P3(IAL)_Summer_2023_Q1

Solution

1. Existence of a root in the interval $[3, 4]$

To show that the function $g(x) = x^6 + 2x - 1000$ has a root α in the interval $[3, 4]$, we apply the **Intermediate Value Theorem**. Since $g(x)$ is a polynomial, it is continuous for all $x \in \mathbb{R}$.

- Evaluate $g(x)$ at the lower bound $x = 3$:

$$\begin{aligned} g(3) &= 3^6 + 2(3) - 1000 \\ &= 729 + 6 - 1000 \\ &= -265 \end{aligned}$$

- Evaluate $g(x)$ at the upper bound $x = 4$:

$$\begin{aligned} g(4) &= 4^6 + 2(4) - 1000 \\ &= 4096 + 8 - 1000 \\ &= 3104 \end{aligned}$$

Since $g(3) < 0$ and $g(4) > 0$, there is a **sign change** across the interval. By the **Intermediate Value Theorem**, there must exist at least one value $\alpha \in (3, 4)$ such that $g(\alpha) = 0$.

2. Iteration to find x_2

The given **iteration formula** is:

$$x_{n+1} = \sqrt[6]{1000 - 2x_n}$$

Given the initial value $x_1 = 3$, we calculate x_2:

$$\begin{aligned} x_2 &= \sqrt[6]{1000 - 2(3)} \\ &= \sqrt[6]{994} \\ &\approx 3.144707... \end{aligned}$$

Rounding to 4 decimal places, we obtain:

$$\boxed{x_2 = 3.1447}$$

3. Repeated iteration to find α

We continue the **fixed-point iteration** process until the values converge to 4 decimal places.

- $x_1 = 3$
- $x_2 = \sqrt[6]{1000 - 2(3)} \approx 3.144707$
- $x_3 = \sqrt[6]{1000 - 2(3.144707)} \approx 3.143615$
- $x_4 = \sqrt[6]{1000 - 2(3.143615)} \approx 3.143623$
- $x_5 = \sqrt[6]{1000 - 2(3.143623)} \approx 3.143623$

The values have stabilized to 4 decimal places. To verify the root α is correct to 4 decimal places, we check the sign of $g(x)$ at the bounds of the rounding interval $[3.14355, 3.14365]$: - $g(3.14355) \approx -0.139$ - $g(3.14365) \approx 0.051$

The sign change confirms the root α lies within this range.

$$\boxed{\alpha = 3.1436}$$

WMA11_P3(IAL)_Summer_2023_Q2

Solution

The problem provides a graph showing a linear relationship between $\log_6 T$ and $\log_6 x$. The line passes through the points $(0, 4)$ and $(2, 0)$.

1. Equation linking $\log_6 T$ and $\log_6 x$

To find the equation of the line, we use the **slope-intercept form** $Y = mX + c$, where $Y = \log_6 T$ and $X = \log_6 x$.

- First, calculate the gradient (m):

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0-4}{2-0} \\ &= -2 \end{aligned}$$

- The y-intercept (c) is given by the point $(0, 4)$, so $c = 4$.

Substituting these values into the linear equation:

$$\log_6 T = -2\log_6 x + 4$$

2. Exact value of T when $x = 216$

- First, determine the value of $\log_6 x$ when $x = 216$:

$$\begin{aligned} \log_6 216 &= \log_6(6^3) \\ &= 3 \end{aligned}$$

- Substitute $\log_6 x = 3$ into the equation found in part (a)(i):

$$\begin{aligned} \log_6 T &= -2(3) + 4 \\ &= -6 + 4 \\ &= -2 \end{aligned}$$

- Solve for T using the definition of a **logarithm**:

$$\begin{aligned} T &= 6^{-2} \\ &= \frac{1}{6^2} \\ &= \frac{1}{36} \end{aligned}$$

3. Equation linking T and x without logarithms

We use **logarithmic properties** to transform the linear equation into an exponential or power form.

$$\log_6 T = -2\log_6 x + 4$$
$$\log_6 T = \log_6(x^{-2}) + \log_6(6^4)$$
$$\log_6 T = \log_6(6^4 \cdot x^{-2})$$

By removing the logarithms from both sides:

$$T = 6^4 \cdot x^{-2}$$
$$T = \frac{1296}{x^2}$$

Final Answers:

(a) (i) $\boxed{\log_6 T = -2\log_6 x + 4}$

(b) (ii) $\boxed{T = \frac{1}{36}}$

(c) $\boxed{T = \frac{1296}{x^2}}$

WMA11_P3(IAL)_Summer_2023_Q3

Solution

1. Differentiation of the logarithmic function

To find the derivative of $f(x) = \ln(\sin^2 3x)$, we first simplify the expression using the **logarithm power rule**, $\ln(a^b) = b \ln a$:

$$\ln(\sin^2 3x) = 2\ln(\sin 3x)$$

Now, we apply the **chain rule** to differentiate with respect to x:

$$\begin{aligned}\frac{d}{dx}[2\ln(\sin 3x)] &= 2 \cdot \frac{1}{\sin 3x} \cdot \frac{d}{dx}(\sin 3x) \\ &= \frac{2}{\sin 3x} \cdot (\cos 3x \cdot 3) \\ &= 6 \cdot \frac{\cos 3x}{\sin 3x}\end{aligned}$$

Using the trigonometric identity $\cot\theta = \frac{\cos\theta}{\sin\theta}$, the simplest form is:

$$\boxed{6\cot 3x}$$

2. Differentiation and Integration of the polynomial power

- **(a) Differentiation** We use the **chain rule** to differentiate $(3x^2-4)^6$:

$$\begin{aligned}\frac{d}{dx}(3x^2-4)^6 &= 6(3x^2-4)^5 \cdot \frac{d}{dx}(3x^2-4) \\ &= 6(3x^2-4)^5 \cdot (6x) \\ &= 36x(3x^2-4)^5\end{aligned}$$

$$\boxed{36x(3x^2-4)^5}$$

- **(b) Integration** The problem asks to show that $\int_0^{\sqrt{2}} x(3x^2-4)^5 dx = R$. From the result in part (a), we know that:

$$\frac{d}{dx}\left[\frac{1}{36}(3x^2-4)^6\right] = x(3x^2-4)^5$$

  By the **Fundamental Theorem of Calculus**, the integral is:

$$\begin{aligned}\int_0^{\sqrt{2}} x(3x^2-4)^5 dx &= \left[\frac{1}{36}(3x^2-4)^6\right]_0^{\sqrt{2}} \\ &= \frac{1}{36}\left[(3(\sqrt{2})^2-4)^6 - (3(0)^2-4)^6\right] \\ &= \frac{1}{36}\left[(3\cdot 2-4)^6 - (-4)^6\right] \\ &= \frac{1}{36}\left[(2)^6 - (4)^6\right]\end{aligned}$$

Calculating the powers:

- $2^6 = 64$
- $4^6 = (2^2)^6 = 2^{12} = 4096$

Substituting these values:

$$R = \frac{64 - 4096}{36}$$
$$= \frac{-4032}{36}$$
$$= -112$$

Thus, the integer R is:

$$\boxed{R = -112}$$

WMA11_P3(IAL)_Summer_2023_Q4

Solution

The function is defined as $f(x) = 2x^2 - 5$ for $x \geq 0$, where $x \in \mathbb{R}$.

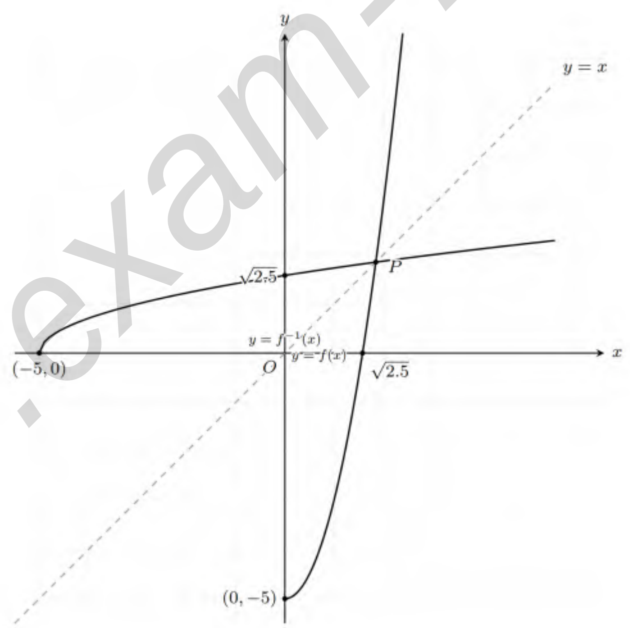
1. State the range of f The **range** of a function is the set of all possible output values. Since the **domain** is $x \geq 0$, we evaluate the function at the boundary and consider its behavior:

- At $x = 0$: $f(0) = 2(0)^2 - 5 = -5$.
- As $x \to \infty$, $f(x) \to \infty$. Since $f(x)$ is a strictly increasing function for $x \geq 0$, the minimum value is -5.

$$\text{Range: } f(x) \geq -5$$

2. Sketch the curve with equation $y = f^{-1}(x)$ The graph of an **inverse function** $y = f^{-1}(x)$ is the reflection of the graph $y = f(x)$ in the line $y = x$.

- The point $(0, -5)$ on f maps to $(-5, 0)$ on f^{-1}.
- The x-intercept of f (where $2x^2 - 5 = 0 \implies x = \sqrt{2.5}$) maps to the y-intercept of f^{-1}.



3. Find the exact x-coordinate of P The point P is the intersection of $y = f(x)$ and $y = f^{-1}(x)$. Since the inverse is a reflection across $y = x$, the intersection of a function and its inverse (for an increasing function) must lie on the line $y = x$. Therefore, we solve $f(x) = x$:

$$2x^2 - 5 = x$$
$$2x^2 - x - 5 = 0$$

Using the **quadratic formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 2$, $b = -1$, and $c = -5$:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-5)}}{2(2)}$$
$$= \frac{1 \pm \sqrt{1+40}}{4}$$
$$= \frac{1 \pm \sqrt{41}}{4}$$

Since the domain of f is $x \geq 0$, we must choose the positive root:

$$x = \frac{1 + \sqrt{41}}{4}$$

$$\boxed{x = \frac{1 + \sqrt{41}}{4}}$$

WMA11_P3(IAL)_Summer_2023_Q5

Solution

1. Solving the first equation

The given equation is $(x-2)(\sqrt{3}\sec x + 2) = 0$ for the interval $0 < x < \pi$. By the **zero-product property**, we set each factor to zero:

- **Case 1: $x - 2 = 0$**

$$x = 2$$

  Since $\pi \approx 3.14159$, the value $x = 2$ lies within the interval $(0, \pi)$.

- **Case 2: $\sqrt{3}\sec x + 2 = 0$**

$$\sqrt{3}\sec x = -2$$
$$\sec x = -\frac{2}{\sqrt{3}}$$

  Using the reciprocal identity $\sec x = \frac{1}{\cos x}$:

$$\frac{1}{\cos x} = -\frac{2}{\sqrt{3}}$$
$$\cos x = -\frac{\sqrt{3}}{2}$$

  In the interval $0 < x < \pi$, the cosine function is negative in the second quadrant. The **reference angle** is $\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

$$x = \pi - \frac{\pi}{6}$$
$$= \frac{5\pi}{6}$$

The solutions for part (i) are $x = 2$ and $x = \frac{5\pi}{6}$.

2. Solving the second equation

The given equation is $10\sin\theta = 3\cos 2\theta$ for $0^\circ < \theta < 360^\circ$. We use the **double-angle identity** for cosine, $\cos 2\theta = 1 - 2\sin^2\theta$, to express the equation in terms of $\sin\theta$:

- **Substitution and Rearrangement:**

$$10\sin\theta = 3(1 - 2\sin^2\theta)$$
$$10\sin\theta = 3 - 6\sin^2\theta$$
$$6\sin^2\theta + 10\sin\theta - 3 = 0$$

- **Solving the Quadratic Equation:** Let $u = \sin\theta$. The equation becomes $6u^2 + 10u - 3 = 0$. Using the **quadratic formula**:

$$u = \frac{-10 \pm \sqrt{10^2 - 4(6)(-3)}}{2(6)}$$
$$= \frac{-10 \pm \sqrt{100 + 72}}{12}$$
$$= \frac{-10 \pm \sqrt{172}}{12}$$
$$= \frac{-10 \pm 2\sqrt{43}}{12}$$
$$= \frac{-5 \pm \sqrt{43}}{6}$$

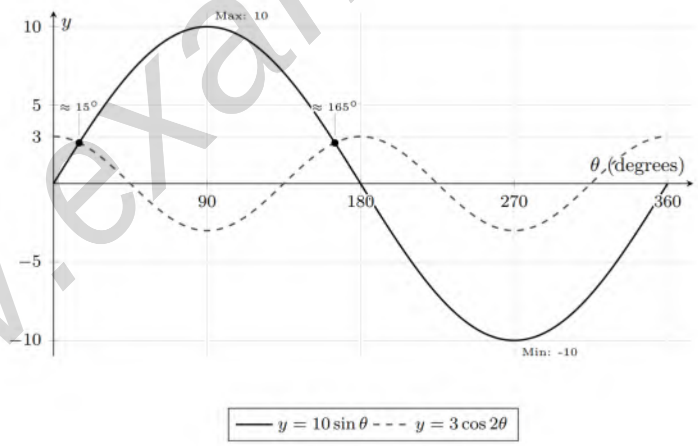
Calculating the decimal values:

$$u_1 = \frac{-5 + \sqrt{43}}{6} \approx 0.259577$$
$$u_2 = \frac{-5 - \sqrt{43}}{6} \approx -1.926244$$

Since the range of the sine function is $[-1, 1]$, u_2 is rejected. We solve for $\sin\theta = 0.259577...$:

- $\theta_1 = \arcsin(0.259577...) \approx 15.045°$
- $\theta_2 = 180° - 15.045° \approx 164.955°$

Rounding to one decimal place as is standard for degrees: $\theta \approx 15.0°, 165.0°$.



Final Answers: (i) $x = 2, \frac{5\pi}{6}$ (ii) $\theta = 15.0°, 165.0°$

WMA11_P3(IAL)_Summer_2023_Q6

Solution

The function is defined as $f(x) = 3 \mid x - 2 \mid - 10$. We will solve the four parts of the problem based on the properties of **absolute value functions**.

1. Coordinates of the vertex P The **vertex** of an absolute value function of the form $y = a \mid x - h \mid + k$ occurs at the point (h, k). For the given function:

- The expression inside the absolute value is zero when $x - 2 = 0$, which gives $x = 2$.
- Substituting $x = 2$ into the function:

$$\begin{aligned} f(2) &= 3 \mid 2 - 2 \mid - 10 \\ &= 3(0) - 10 \\ &= -10 \end{aligned}$$

Thus, the coordinates of P are $(2, -10)$. $\boxed{P(2, -10)}$

2. Finding $ff(0)$ This is a **composite function** $f(f(0))$.

- First, calculate $f(0)$:

$$\begin{aligned} f(0) &= 3 \mid 0 - 2 \mid - 10 \\ &= 3(2) - 10 \\ &= 6 - 10 \\ &= -4 \end{aligned}$$

- Next, calculate $f(-4)$:

$$\begin{aligned} f(-4) &= 3 \mid -4 - 2 \mid - 10 \\ &= 3 \mid -6 \mid - 10 \\ &= 3(6) - 10 \\ &= 18 - 10 \\ &= 8 \end{aligned}$$

$\boxed{ff(0) = 8}$

3. Solving the inequality $3 \mid x - 2 \mid - 10 < 5x + 10$ To solve the inequality, we first find the critical points where $3 \mid x - 2 \mid - 10 = 5x + 10$.

- **Case 1:** $x \geq 2$

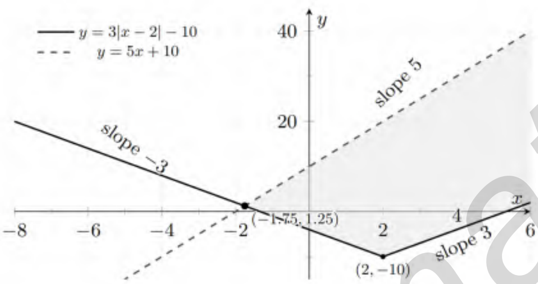
$$\begin{aligned} 3(x - 2) - 10 &= 5x + 10 \\ 3x - 6 - 10 &= 5x + 10 \\ 3x - 16 &= 5x + 10 \\ -2x &= 26 \\ x &= -13 \end{aligned}$$

Since $-13 < 2$, this solution is invalid for this case.

- **Case 2:** $x < 2$

$$3(-(x-2)) - 10 = 5x + 10$$
$$-3x + 6 - 10 = 5x + 10$$
$$-3x - 4 = 5x + 10$$
$$-8x = 14$$
$$x = -1.75$$

Since $-1.75 < 2$, this is a valid intersection point.



Checking the boundary $x = -1.75$: If $x = 0$ (which is > -1.75): $f(0) = -4$ and $5(0) + 10 = 10$. Since $-4 < 10$, the inequality holds for $x > -1.75$. $\boxed{x > -1.75}$

4. Solving the equation $f(\mid x \mid) = 0$ The equation is $3 \mid \mid x \mid - 2 \mid - 10 = 0$.

$$3 \mid \mid x \mid - 2 \mid = 10$$
$$\mid \mid x \mid - 2 \mid = \frac{10}{3}$$

This leads to two sub-cases for the outer absolute value:

- **Sub-case A:** $\mid x \mid - 2 = \frac{10}{3}$

$$\mid x \mid = \frac{10}{3} + 2$$
$$\mid x \mid = \frac{16}{3} \Longrightarrow x = \pm\frac{16}{3}$$

- **Sub-case B:** $\mid x \mid - 2 = -\frac{10}{3}$

$$\mid x \mid = 2 - \frac{10}{3}$$
$$\mid x \mid = -\frac{4}{3}$$

Since the absolute value of a real number cannot be negative, there are no solutions from Sub-case B. $\boxed{x = \pm\frac{16}{3}}$

WMA11_P3(IAL)_Summer_2023_Q7

Solution

1. Analysis of the First Population

The growth of the first population is governed by the **exponential growth** model:

$$N_1 = Ae^{kt}$$

where t is the time in hours and N_1 is the number of bacteria.

- **Finding the value of A:** At the start of the study ($t = 0$), the population is given as 2500.

$$\begin{aligned} 2500 &= Ae^{k(0)} \\ 2500 &= A \cdot 1 \\ A &= 2500 \end{aligned}$$

- **Finding the value of k:** At $t = 8$, the population is 10000. Substituting $A = 2500$:

$$\begin{aligned} 10000 &= 2500e^{8k} \\ \frac{10000}{2500} &= e^{8k} \\ 4 &= e^{8k} \\ \ln(4) &= 8k \\ k &= \frac{\ln(4)}{8} \end{aligned}$$

  Calculating the numerical value to 4 significant figures:

$$k \approx 0.1733$$

2. Analysis of the Second Population

The second population is modeled by:

$$N_2 = 60000e^{-0.6t}$$

- **Finding the rate of decrease at $t = 5$:** The **rate of change** is found by taking the first derivative of N_2 with respect to t:

$$\begin{aligned} \frac{dN_2}{dt} &= \frac{d}{dt}(60000e^{-0.6t}) \\ &= 60000 \cdot (-0.6) \cdot e^{-0.6t} \\ &= -36000e^{-0.6t} \end{aligned}$$

  At $t = 5$:

$$\begin{aligned} \frac{dN_2}{dt}\Big|_{t=5} &= -36000e^{-0.6(5)} \\ &= -36000e^{-3} \\ &\approx -1792.239 \end{aligned}$$

  The rate of decrease is the magnitude of this value. To 3 significant figures:

$$\text{Rate of decrease} \approx 1790\,\text{bacteria} \cdot \text{h}^{-1}$$

3. Finding the Time T when Populations are Equal

We set $N_1 = N_2$ at $t = T$:

$$2500e^{kT} = 60000e^{-0.6T}$$

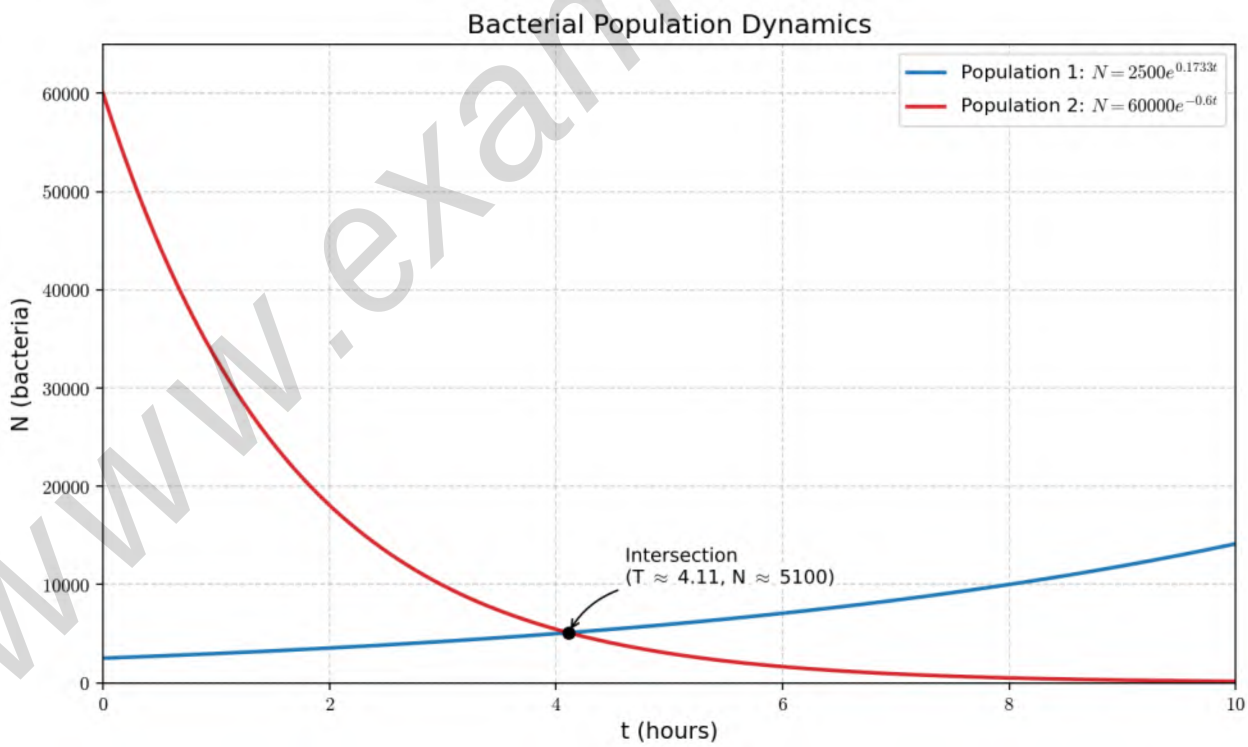
Using the exact value $k = \frac{\ln(4)}{8}$:

$$\frac{e^{kT}}{e^{-0.6T}} = \frac{60000}{2500}$$
$$e^{(k+0.6)T} = 24$$
$$(k+0.6)T = \ln(24)$$
$$T = \frac{\ln(24)}{k+0.6}$$
$$T = \frac{\ln(24)}{\frac{\ln(4)}{8} + 0.6}$$

Calculating the numerical value:

$$T \approx \frac{3.17805}{0.173287 + 0.6}$$
$$T \approx \frac{3.17805}{0.773287}$$
$$T \approx 4.10979$$

To 3 significant figures, $T = 4.11$.



(a) $A = 2500$, $k = 0.1733$
(b) $1790\,\text{bacteria} \cdot \text{h}^{-1}$
(c) $T = 4.11$

WMA11_P3(IAL)_Summer_2023_Q8

Solution

The problem involves the analysis of the function $f(x) = (2x+1)^3 e^{-4x}$ and its transformation $g(x) = 8f(x-2)$.

1. Differentiation of $f(x)$

To find the derivative $f'(x)$, we apply the **product rule** and the **chain rule**. Let $u(x) = (2x+1)^3$ and $v(x) = e^{-4x}$.

- The derivative of $u(x)$ is $u'(x) = 3(2x+1)^2 \cdot 2 = 6(2x+1)^2$.
- The derivative of $v(x)$ is $v'(x) = -4e^{-4x}$.

Applying the product rule $f'(x) = u'v + uv'$:

$$\begin{aligned} f'(x) &= 6(2x+1)^2 e^{-4x} + (2x+1)^3(-4e^{-4x}) \\ &= 2(2x+1)^2 e^{-4x}[3 - 2(2x+1)] \\ &= 2(2x+1)^2 e^{-4x}(3 - 4x - 2) \\ &= 2(2x+1)^2(1-4x)e^{-4x} \end{aligned}$$

Comparing this to the form $f'(x) = A(2x+1)^2(1-4x)e^{-4x}$, we find:

$$\boxed{A = 2}$$

2. Stationary points on C

Stationary points occur where $f'(x) = 0$. Since $e^{-4x} \neq 0$ for all x, we solve:

$$2(2x+1)^2(1-4x) = 0$$

- From $(2x+1)^2 = 0$, we get $x = -0.5$.
- From $(1-4x) = 0$, we get $x = 0.25$.

Now, calculate the corresponding y-coordinates:

- For $x = -0.5$:

  $f(-0.5) = (2(-0.5)+1)^3 e^{-4(-0.5)} = 0^3 \cdot e^2 = 0$

- For $x = 0.25$:

  $f(0.25) = (2(0.25)+1)^3 e^{-4(0.25)} = (1.5)^3 e^{-1} = \frac{27}{8e}$

The exact coordinates of the stationary points are:

$$\boxed{(-0.5, 0) \quad \text{and} \quad (0.25, \frac{27}{8e})}$$

3. Maximum stationary point of $g(x)$

The function $g(x) = 8f(x-2)$ represents a **transformation** of $f(x)$ consisting of:

- A horizontal translation of 2 units to the right.
- A vertical stretch by a factor of 8.

From the sketch of $f(x)$ and the coordinates found, the point $(0.25, \frac{27}{8e})$ is the local maximum, while $(-0.5, 0)$ is a point of inflection (or minimum depending on the global context, but clearly not the peak).

Applying the transformations to the maximum point $(x_m, y_m) = (0.25, \frac{27}{8e})$:

- New x-coordinate: $x' = x_m + 2 = 0.25 + 2 = 2.25$.
- New y-coordinate: $y' = 8 \cdot y_m = 8 \cdot \frac{27}{8e} = \frac{27}{e}$.

The coordinates of the maximum stationary point on $y = g(x)$ are:

$$\boxed{(2.25, \frac{27}{e})}$$

WMA11_P3(IAL)_Summer_2023_Q9

Solution

1. Verification of the Trigonometric Identity

To show that $\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x} \equiv \csc x$, we begin by expressing the left-hand side (LHS) over a common denominator:

$$\text{LHS} = \frac{\cos 2x \cos x + \sin 2x \sin x}{\sin x \cos x}$$

- Applying the **cosine difference identity**, $\cos(A - B) = \cos A \cos B + \sin A \sin B$, where $A = 2x$ and $B = x$:

$$\begin{aligned}\text{LHS} &= \frac{\cos(2x - x)}{\sin x \cos x} \\ &= \frac{\cos x}{\sin x \cos x}\end{aligned}$$

- Given $x \neq \frac{n\pi}{2}$, $\cos x \neq 0$, we can simplify the fraction:

$$\begin{aligned}\text{LHS} &= \frac{1}{\sin x} \\ &= \csc x = \text{RHS}\end{aligned}$$

2. Solving the Equation for θ

Using the identity derived in part (a), the equation $\left(\frac{\cos 2\theta}{\sin \theta} + \frac{\sin 2\theta}{\cos \theta}\right)^2 = 6 \cot \theta - 4$ simplifies to:

$$\begin{aligned}(\csc \theta)^2 &= 6 \cot \theta - 4 \\ \csc^2 \theta &= 6 \cot \theta - 4\end{aligned}$$

- We use the **Pythagorean identity** $\csc^2 \theta = 1 + \cot^2 \theta$ to transform the equation into a quadratic in terms of $\cot \theta$:

$$\begin{aligned}1 + \cot^2 \theta &= 6 \cot \theta - 4 \\ \cot^2 \theta - 6 \cot \theta + 5 &= 0\end{aligned}$$

- Factoring the quadratic expression:

$$(\cot \theta - 1)(\cot \theta - 5) = 0$$

- This yields two possible values for $\cot \theta$:

1. $\cot \theta = 1 \Longrightarrow \tan \theta = 1$
2. $\cot \theta = 5 \Longrightarrow \tan \theta = 0.2$

- Solving for θ in the interval $0 < \theta < \frac{\pi}{2}$:
  - For $\tan \theta = 1$: $\theta = \arctan(1) = \frac{\pi}{4} \approx 0.785$
  - For $\tan \theta = 0.2$: $\theta = \arctan(0.2) \approx 0.197$

Rounding to 3 significant figures, we obtain: $\boxed{\theta = 0.785, 0.197}$

3. Evaluation of the Definite Integral

We evaluate the integral $I = \int_{\pi/6}^{\pi/4} \left(\frac{\cos 2x}{\sin x} + \frac{\sin 2x}{\cos x}\right) \cot x \, dx$.

- Substituting the identity from part (a):

$$I = \int_{\pi/6}^{\pi/4} \csc x \cot x \, dx$$

- The **antiderivative** of $\csc x \cot x$ is $-\csc x$:

$$\begin{aligned} I &= [-\csc x]_{\pi/6}^{\pi/4} \\ &= -\csc\left(\frac{\pi}{4}\right) - \left(-\csc\left(\frac{\pi}{6}\right)\right) \\ &= -\sqrt{2} + 2 \end{aligned}$$

The exact value of the integral is: $\boxed{2 - \sqrt{2}}$

WMA11_P3(IAL)_Summer_2023_Q10

Solution

The problem involves finding the derivative of a rational function and using it to determine the equations of vertical tangents to a curve.

1. Finding the derivative dx/dy

The equation of the curve is given by:

$$x = \frac{2y^2 + 6}{3y - 3}$$

To find $\frac{dx}{dy}$, we apply the **quotient rule**, which states that for a function $x = \frac{u}{v}$, the derivative is $\frac{dx}{dy} = \frac{v\frac{du}{dy} - u\frac{dv}{dy}}{v^2}$. Let $u = 2y^2 + 6$ and $v = 3y - 3$. Then:

- $\frac{du}{dy} = 4y$
- $\frac{dv}{dy} = 3$

Substituting these into the quotient rule formula:

$$\begin{aligned}\frac{dx}{dy} &= \frac{(3y-3)(4y) - (2y^2+6)(3)}{(3y-3)^2} \\ &= \frac{12y^2 - 12y - (6y^2 + 18)}{(3y-3)^2} \\ &= \frac{12y^2 - 12y - 6y^2 - 18}{(3y-3)^2} \\ &= \frac{6y^2 - 12y - 18}{(3y-3)^2}\end{aligned}$$

To simplify the fraction, we factor the numerator and the denominator:

- Numerator: $6(y^2 - 2y - 3) = 6(y-3)(y+1)$
- Denominator: $(3(y-1))^2 = 9(y-1)^2$

$$\begin{aligned}\frac{dx}{dy} &= \frac{6(y-3)(y+1)}{9(y-1)^2} \\ &= \frac{2(y-3)(y+1)}{3(y-1)^2}\end{aligned}$$

Expanding the numerator back for the final simplified form:

$$\frac{dx}{dy} = \frac{2y^2 - 4y - 6}{3(y-1)^2}$$

2. Finding the equations of the vertical tangents

The tangents at points P and Q are parallel to the y-axis. A **vertical tangent** occurs where the gradient $\frac{dy}{dx}$ is undefined, which corresponds to the condition:

$$\frac{dx}{dy} = 0$$

Using the simplified derivative from part (a):

$$\frac{2(y-3)(y+1)}{3(y-1)^2} = 0$$
$$2(y-3)(y+1) = 0$$

This gives two y-coordinates for the points of tangency:

- $y_1 = 3$
- $y_2 = -1$

Now, we find the corresponding x-coordinates by substituting these values back into the original equation $x = \frac{2y^2+6}{3y-3}$.

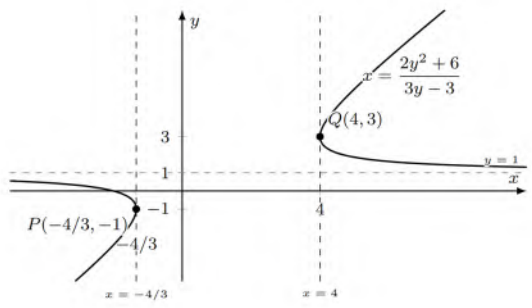
- For $y = 3$:

$$\begin{aligned} x &= \frac{2(3)^2+6}{3(3)-3} \\ &= \frac{18+6}{9-3} \\ &= \frac{24}{6} = 4 \end{aligned}$$

- For $y = -1$:

$$\begin{aligned} x &= \frac{2(-1)^2+6}{3(-1)-3} \\ &= \frac{2+6}{-3-3} \\ &= \frac{8}{-6} = -\frac{4}{3} \end{aligned}$$

The equations of the tangents are vertical lines of the form $x = c$. Based on the sketch in Figure 4, point P is on the left ($x < 0$) and point Q is on the right ($x > 0$).



The equations of the two tangents are:

$$\boxed{x = 4 \quad \text{and} \quad x = -\frac{4}{3}}$$

WMA11_P3(IAL)_Winter_2023_Q1

Solution

The functions f and g are defined as follows:

$$f(x) = 9 - x^2, \quad x \in \mathbb{R}, x \geq 0$$
$$g(x) = \frac{3}{2x+1}, \quad x \in \mathbb{R}, x \geq 0$$

1. Range of f The **range** of a function is the set of all possible output values.

- For $f(x) = 9 - x^2$ with the **domain** $x \geq 0$:
- When $x = 0$, $f(0) = 9 - 0^2 = 9$.
- As x increases from 0 toward infinity, x^2 increases, which causes $f(x) = 9 - x^2$ to decrease.
- Since $x^2 \geq 0$ for all x, the maximum value of $f(x)$ is 9. There is no lower bound as $x \to \infty$, but the function is typically considered over its real values.
- Thus, the range is $f(x) \leq 9$.

$\boxed{f(x) \leq 9}$

2. Value of $fg(1.5)$ To find the value of the **composite function** $fg(1.5)$, we first evaluate $g(1.5)$ and then substitute that result into f.

- Step 1: Calculate $g(1.5)$

$$\begin{aligned} g(1.5) &= \frac{3}{2(1.5)+1} \\ &= \frac{3}{3+1} \\ &= \frac{3}{4} = 0.75 \end{aligned}$$

- Step 2: Calculate $f(g(1.5)) = f(0.75)$

$$\begin{aligned} f(0.75) &= 9 - (0.75)^2 \\ &= 9 - 0.5625 \\ &= 8.4375 \end{aligned}$$

$\boxed{8.4375}$

3. Find g^{-1} To find the **inverse function** $g^{-1}(x)$, we set $y = g(x)$ and solve for x in terms of y.

- Start with the equation:

$$y = \frac{3}{2x+1}$$

- Multiply both sides by $(2x+1)$:

$$y(2x+1) = 3$$

- Expand and isolate the term with x:

$$\begin{aligned} 2xy + y &= 3 \\ 2xy &= 3 - y \\ x &= \frac{3-y}{2y} \end{aligned}$$

- Swap x and y to write the inverse function:

$$g^{-1}(x) = \frac{3-x}{2x}$$

- The domain of g^{-1} is the range of g. Since $x \geq 0$, $g(0) = 3$ and $\lim_{x\to\infty} g(x) = 0$. Thus, the domain of g^{-1} is $0 < x \leq 3$.

$$\boxed{g^{-1}(x) = \frac{3-x}{2x}}$$

WMA11_P3(IAL)_Winter_2023_Q2

Solution

1. Expressing $f(x)$ in the form $R\cos(x-\alpha)$

To express $f(x) = \cos x + 2\sin x$ in the form $R\cos(x-\alpha)$, we use the **harmonic addition theorem** (or **R-formula**). We expand the target expression using the **cosine addition formula**:

$$R\cos(x-\alpha) = R(\cos x\cos\alpha + \sin x\sin\alpha) = (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$$

- **Step 1: Determine R** By comparing coefficients with $f(x) = 1\cdot\cos x + 2\cdot\sin x$, we have:

$$R\cos\alpha = 1$$
$$R\sin\alpha = 2$$

Squaring and adding these equations:

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 2^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 1 + 4$$
$$R^2 = 5$$
$$R = \sqrt{5} \quad (\text{since} \quad R > 0)$$

- **Step 2: Determine α** Dividing the sine equation by the cosine equation:

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{1}$$
$$\tan\alpha = 2$$
$$\alpha = \arctan(2)$$

Since $0 < \alpha < \frac{\pi}{2}$ and both $\sin\alpha$ and $\cos\alpha$ are positive, α lies in the first quadrant. Calculating the value in radians:

$$\alpha \approx 1.1071487...$$

Rounding to 3 decimal places: $\alpha = 1.107\,\text{rad}$.

Thus, $f(x) = \sqrt{5}\cos(x - 1.107)$.

2. Analyzing $g(x) = 3 - 7f(2x)$

Using the result from part (a), we substitute $2x$ into the expression for $f(x)$:

$$g(x) = 3 - 7[\sqrt{5}\cos(2x-\alpha)] = 3 - 7\sqrt{5}\cos(2x-\alpha)$$

- **(i) Exact maximum value of $g(x)$** The function $g(x)$ is a **linear transformation** of a cosine wave. The cosine function $\cos(2x-\alpha)$ oscillates between -1 and 1. To maximize $g(x) = 3 - 7\sqrt{5}\cos(2x-\alpha)$, we need the term being subtracted to be as small as possible. This occurs when $\cos(2x-\alpha) = -1$.

$$g_{\max} = 3 - 7\sqrt{5}(-1)$$
$$= 3 + 7\sqrt{5}$$

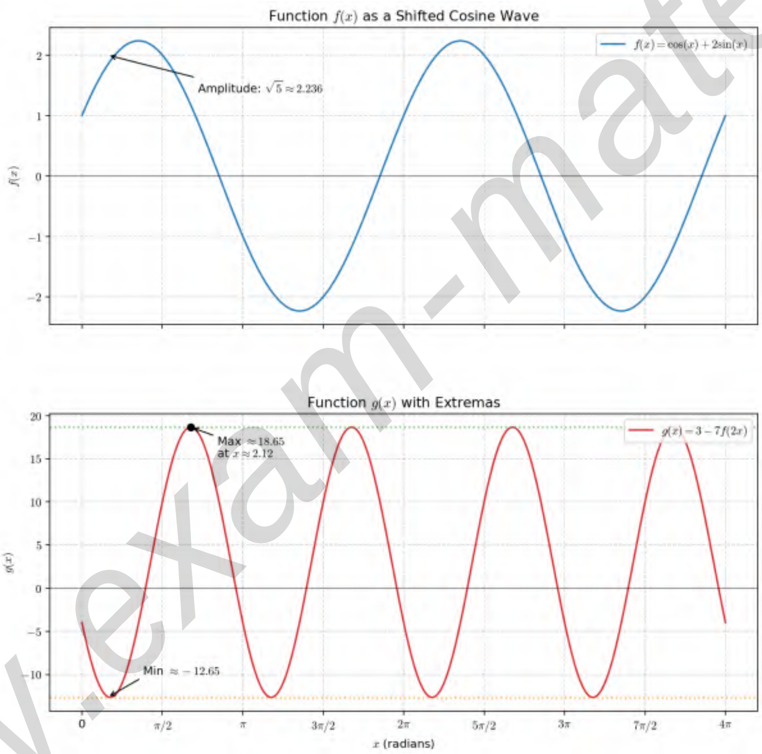
- **(ii) Smallest positive value of x for the maximum** The maximum occurs when $\cos(2x - \alpha) = -1$. The general solution for $\cos\theta = -1$ is $\theta = \pi, 3\pi, 5\pi, ...$ (or $(2n+1)\pi$ for integer n). Setting the argument equal to π:

$$2x - \alpha = \pi$$
$$2x = \pi + \alpha$$
$$x = \frac{\pi + \alpha}{2}$$

Using the unrounded value of $\alpha = \arctan(2)$:

$$x = \frac{\pi + 1.1071487...}{2}$$
$$= \frac{4.248741...}{2}$$
$$= 2.12437...$$

Rounding to 2 decimal places: $x = 2.12\,\mathrm{rad}$.



Final Answers:

(a) $R = \sqrt{5}$, $\alpha = 1.107$
(b) (i) $\text{Maximum} = 3 + 7\sqrt{5}$
(c) (ii) $x = 2.12$

$\boxed{R = \sqrt{5}, \alpha = 1.107}$ $\boxed{\text{(i)} \quad 3 + 7\sqrt{5}, \ \text{(ii)} \quad 2.12}$

WMA11_P3(IAL)_Winter_2023_Q3

Solution

1. Equation of the line l

The graph displays a **linear relationship** between the variables $X = x$ and $Y = \log_{10} y$. The line l passes through the points $(-4.8, 0)$ and $(0, 1.5)$.

- The **gradient** m of the line is calculated as:

$$\begin{aligned} m &= \frac{Y_2 - Y_1}{x_2 - x_1} \\ &= \frac{1.5 - 0}{0 - (-4.8)} \\ &= \frac{1.5}{4.8} \\ &= 0.3125 \end{aligned}$$

- The Y-intercept c is given directly by the point $(0, 1.5)$, so $c = 1.5$.
- Using the **slope-intercept form** $Y = mx + c$, the equation for l is:

$$\log_{10} y = 0.3125x + 1.5$$

2. Expressing y in the form kb^x

To find the relationship between y and x, we apply the inverse of the **common logarithm** (base 10) to both sides of the equation found in part (a).

$$\begin{aligned} y &= 10^{0.3125x+1.5} \\ &= 10^{1.5} \cdot 10^{0.3125x} \\ &= 10^{1.5} \cdot (10^{0.3125})^x \end{aligned}$$

Comparing this to the form $y = kb^x$, we identify the constants k and b: - $k = 10^{1.5}$ - $b = 10^{0.3125}$

Calculating the numerical values to 3 significant figures:

$$\begin{aligned} k &= 10^{1.5} \approx 31.6227766... \approx 31.6 \\ b &= 10^{0.3125} \approx 2.053525... \approx 2.05 \end{aligned}$$

The final expression for y is:

$$y = 31.6 \cdot 2.05^x$$

Final Answer: (a) $\boxed{\log_{10} y = 0.3125x + 1.5}$ (b) $\boxed{y = 31.6 \cdot 2.05^x \quad \text{where} \quad k = 31.6, b = 2.05}$

WMA11_P3(IAL)_Winter_2023_Q4

Solution

The problem asks to decompose a rational function $f(x)$ into a polynomial and a remainder term, and then integrate the resulting expression.

1. Finding the constants A, B, C, and D

To find the constants, we perform **polynomial long division** or use the method of equating coefficients. The given function is:

$$f(x) = \frac{2x^4 + 15x^3 + 35x^2 + 21x - 4}{(x+3)^2}$$

First, expand the denominator:

$$(x+3)^2 = x^2 + 6x + 9$$

We divide the numerator $2x^4 + 15x^3 + 35x^2 + 21x - 4$ by $x^2 + 6x + 9$:

- **Step 1:** Divide the leading terms: $\frac{2x^4}{x^2} = 2x^2$. Multiply $2x^2(x^2 + 6x + 9) = 2x^4 + 12x^3 + 18x^2$. Subtract from the numerator:

$$(2x^4 + 15x^3 + 35x^2 + 21x - 4) - (2x^4 + 12x^3 + 18x^2) = 3x^3 + 17x^2 + 21x - 4$$

- **Step 2:** Divide the new leading term: $\frac{3x^3}{x^2} = 3x$. Multiply $3x(x^2 + 6x + 9) = 3x^3 + 18x^2 + 27x$. Subtract:

$$(3x^3 + 17x^2 + 21x - 4) - (3x^3 + 18x^2 + 27x) = -x^2 - 6x - 4$$

- **Step 3:** Divide the new leading term: $\frac{-x^2}{x^2} = -1$. Multiply $-1(x^2 + 6x + 9) = -x^2 - 6x - 9$. Subtract:

$$(-x^2 - 6x - 4) - (-x^2 - 6x - 9) = 5$$

The quotient is $2x^2 + 3x - 1$ and the remainder is 5. Thus:

$$f(x) = 2x^2 + 3x - 1 + \frac{5}{(x+3)^2}$$

Comparing this to the form $f(x) = Ax^2 + Bx + C + \frac{D}{(x+3)^2}$, we identify:

$$\boxed{A = 2, B = 3, C = -1, D = 5}$$

2. Integrating the function $f(x)$

Using the decomposed form found in part (a), we integrate term by term:

$$\int f(x)\,dx = \int \left(2x^2 + 3x - 1 + \frac{5}{(x+3)^2}\right) dx$$

Applying the **power rule for integration** $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ and the substitution rule for the last term:

- **Polynomial terms:**

$$\int (2x^2 + 3x - 1)\, dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - x$$

- **Remainder term:** Let $u = x + 3$, then $du = dx$.

$$\int \frac{5}{(x+3)^2}\, dx = 5\int (x+3)^{-2}\, dx = 5\left[\frac{(x+3)^{-1}}{-1}\right] = -\frac{5}{x+3}$$

Combining the results and adding the **constant of integration** K:

$$\int f(x)\, dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - x - \frac{5}{x+3} + K$$

Final Answer: (a) $\boxed{A = 2, B = 3, C = -1, D = 5}$ (b) $\boxed{\int f(x)\, dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - x - \frac{5}{x+3} + K}$

WMA11_P3(IAL)_Winter_2023_Q5

Solution

1. Proof of the Trigonometric Identity

To prove the identity $\cot^2 x - \tan^2 x = 4\cot 2x \csc 2x$, we begin by manipulating the left-hand side (LHS) using the **difference of squares** and basic **trigonometric identities**.

- **Step 1: Factor the expression**

$$\begin{aligned}\text{LHS} &= \cot^2 x - \tan^2 x \\ &= (\cot x - \tan x)(\cot x + \tan x)\end{aligned}$$

- **Step 2: Simplify the factors using sine and cosine** For the first factor:

$$\begin{aligned}\cot x - \tan x &= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} \\ &= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x}\end{aligned}$$

  Applying the **double angle formula** $\cos 2x = \cos^2 x - \sin^2 x$ and $2\sin x \cos x = \sin 2x$:

$$\begin{aligned}\cot x - \tan x &= \frac{\cos 2x}{\frac{1}{2}\sin 2x} \\ &= 2\cot 2x\end{aligned}$$

  For the second factor:

$$\begin{aligned}\cot x + \tan x &= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\ &= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} \\ &= \frac{1}{\frac{1}{2}\sin 2x} \\ &= 2\csc 2x\end{aligned}$$

- **Step 3: Combine the results**

$$\begin{aligned}\text{LHS} &= (2\cot 2x)(2\csc 2x) \\ &= 4\cot 2x \csc 2x\end{aligned}$$

  Since the LHS equals the right-hand side (RHS), the identity is proven.

2. Solving the Equation

We are given the equation $4\cot 2\theta \csc 2\theta = 2\tan^2\theta$ for the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

- **Step 1: Substitute the proven identity** From part (a), we know $4\cot 2\theta \csc 2\theta = \cot^2\theta - \tan^2\theta$. Substituting this into the equation:

$$\cot^2\theta - \tan^2\theta = 2\tan^2\theta$$

- **Step 2: Simplify the equation**

$$\cot^2\theta = 3\tan^2\theta$$
$$\frac{1}{\tan^2\theta} = 3\tan^2\theta$$
$$\tan^4\theta = \frac{1}{3}$$

- **Step 3: Solve for θ** Taking the square root of both sides:

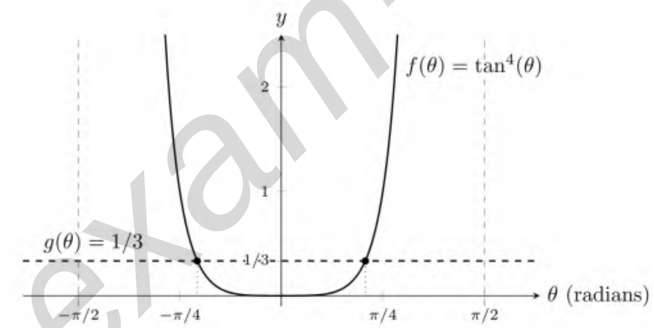
$$\tan^2\theta = \frac{1}{\sqrt{3}}$$

Taking the square root again:

$$\tan\theta = \pm\sqrt{\frac{1}{\sqrt{3}}} = \pm 3^{-1/4}$$

We calculate the values for θ within the range $(-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\theta_1 = \arctan(3^{-1/4}) \approx 0.6505\,\text{rad}$$
$$\theta_2 = \arctan(-3^{-1/4}) \approx -0.6505\,\text{rad}$$



Rounding to 2 decimal places:

$$\theta \approx \pm 0.65$$

$\boxed{\theta = -0.65, 0.65}$

WMA11_P3(IAL)_Winter_2023_Q6

Solution

The problem involves analyzing the properties of an **absolute value function** and solving an equation involving multiple absolute value terms. The given function is:

$$y = |\ 3x - 5a\ | - 2a$$

where $a > 0$.

1. Finding the coordinates of key points

- **(i) Point P (y-intercept)** The **y-intercept** occurs when $x = 0$.

$$\begin{aligned} y &= |\ 3(0) - 5a\ | - 2a \\ &= | - 5a\ | - 2a \end{aligned}$$

  Since $a > 0$, $| - 5a\ | = 5a$.

$$\begin{aligned} y &= 5a - 2a \\ &= 3a \end{aligned}$$

  Thus, the coordinates of P are $(0, 3a)$.

- **(ii) Points Q and R (x-intercepts)** The **x-intercepts** occur when $y = 0$.

$$\begin{aligned} |\ 3x - 5a\ | - 2a &= 0 \\ |\ 3x - 5a\ | &= 2a \end{aligned}$$

  This yields two linear equations:

$$\begin{aligned} 3x - 5a = 2a \quad &\text{or} \quad 3x - 5a = -2a \\ 3x = 7a \quad &\text{or} \quad 3x = 3a \\ x = \frac{7}{3}a \quad &\text{or} \quad x = a \end{aligned}$$

  From the graph, Q is closer to the origin than R. Therefore, Q is $(a, 0)$ and R is $(\frac{7}{3}a, 0)$.

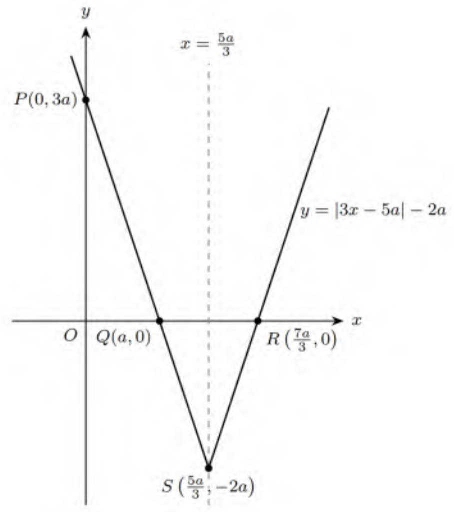
- **(iii) Point S (minimum point)** The **minimum point** of a function of the form $y = |\ f(x)\ | + k$ occurs at the vertex where the expression inside the absolute value is zero.

$$\begin{aligned} 3x - 5a &= 0 \\ x &= \frac{5}{3}a \end{aligned}$$

  Substituting $x = \frac{5}{3}a$ into the equation:

$$\begin{aligned} y &= |\ 3(\frac{5}{3}a) - 5a\ | - 2a \\ &= |\ 5a - 5a\ | - 2a \\ &= -2a \end{aligned}$$

  Thus, the coordinates of S are $(\frac{5}{3}a, -2a)$.



2. Solving the equation $| \; 3x - 5a \; | - 2a = | \; x - 2a \; |$

To solve this equation, we consider the critical values where the expressions inside the absolute values change sign: $x = \frac{5}{3}a$ and $x = 2a$. Since $a > 0$, we have $\frac{5}{3}a < 2a$.

- **Case 1: $x \geq 2a$** Both expressions inside the absolute values are non-negative.

$$(3x - 5a) - 2a = (x - 2a)$$
$$3x - 7a = x - 2a$$
$$2x = 5a$$
$$x = \frac{5}{2}a$$

  Checking the boundary: $\frac{5}{2}a = 2.5a$, which is $\geq 2a$. This is a valid solution.

- **Case 2: $\frac{5}{3}a \leq x < 2a$** The first expression is non-negative, the second is negative.

$$(3x - 5a) - 2a = -(x - 2a)$$
$$3x - 7a = -x + 2a$$
$$4x = 9a$$
$$x = \frac{9}{4}a$$

  Checking the boundary: $\frac{9}{4}a = 2.25a$. This is not in the range $[\frac{5}{3}a, 2a)$, so it is rejected.

- **Case 3: $x < \frac{5}{3}a$** Both expressions are negative (Note: $x - 2a$ is negative because $x < \frac{5}{3}a < 2a$).

$$-(3x - 5a) - 2a = -(x - 2a)$$
$$-3x + 5a - 2a = -x + 2a$$
$$-3x + 3a = -x + 2a$$
$$-2x = -a$$
$$x = \frac{1}{2}a$$

Checking the boundary: $\frac{1}{2}a = 0.5a$, which is $< \frac{5}{3}a \approx 1.67a$. This is a valid solution.

The values of x are:

$$\boxed{x = \frac{1}{2}a, \quad x = \frac{5}{2}a}$$

WMA11_P3(IAL)_Winter_2023_Q7

Solution

The curve C is defined by the equation $x = 3\tan\left(y - \frac{\pi}{6}\right)$ for $x \in \mathbb{R}$ and $-\frac{\pi}{3} < y < \frac{2\pi}{3}$.

1. Differentiation of the Implicit Function

- To find $\frac{dy}{dx}$, we first differentiate x with respect to y using the **chain rule** and the derivative of the tangent function:

$$\frac{dx}{dy} = \frac{d}{dy}\left[3\tan\left(y - \frac{\pi}{6}\right)\right]$$
$$= 3\sec^2\left(y - \frac{\pi}{6}\right)$$

- We use the **trigonometric identity** $\sec^2\theta = 1 + \tan^2\theta$ to express the derivative in terms of x:

$$\frac{dx}{dy} = 3\left[1 + \tan^2\left(y - \frac{\pi}{6}\right)\right]$$

- From the original equation, $\tan\left(y - \frac{\pi}{6}\right) = \frac{x}{3}$. Substituting this into the expression for $\frac{dx}{dy}$:

$$\frac{dx}{dy} = 3\left[1 + \left(\frac{x}{3}\right)^2\right]$$
$$= 3\left(1 + \frac{x^2}{9}\right)$$
$$= 3 + \frac{x^2}{3}$$
$$= \frac{9 + x^2}{3}$$

- Taking the reciprocal to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$
$$= \frac{3}{x^2 + 9}$$

Comparing this to the form $\frac{a}{x^2+b}$, we identify $a = 3$ and $b = 9$.

2. Finding the x-coordinate of Q

- **Step 1: Find the coordinates of point P.** The y-coordinate of P is $y_P = \frac{\pi}{3}$. We find the x-coordinate x_P by substituting y_P into the curve equation:

$$\begin{aligned} x_P &= 3\tan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) \\ &= 3\tan\left(\frac{\pi}{6}\right) \\ &= 3\left(\frac{1}{\sqrt{3}}\right) \\ &= \sqrt{3} \end{aligned}$$

So, $P = (\sqrt{3}, \frac{\pi}{3})$.

- **Step 2: Find the gradient of the tangent at P.** Substitute $x = \sqrt{3}$ into the expression for $\frac{dy}{dx}$:

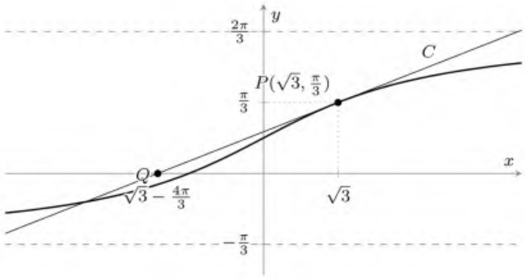
$$\begin{aligned} m_{\text{tangent}} &= \frac{dy}{dx}\Big|_{x=\sqrt{3}} \\ &= \frac{3}{(\sqrt{3})^2 + 9} \\ &= \frac{3}{3+9} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

- **Step 3: Determine the equation of the tangent line.** Using the **point-slope form** $y - y_P = m(x - x_P)$:

$$y - \frac{\pi}{3} = \frac{1}{4}(x - \sqrt{3})$$

- **Step 4: Find the x-coordinate of Q.** Point Q is where the tangent crosses the x-axis, so we set $y = 0$:

$$\begin{aligned} 0 - \frac{\pi}{3} &= \frac{1}{4}(x_Q - \sqrt{3}) \\ -\frac{4\pi}{3} &= x_Q - \sqrt{3} \\ x_Q &= \sqrt{3} - \frac{4\pi}{3} \end{aligned}$$



$$x = \sqrt{3} - \frac{4\pi}{3}$$

WMA11_P3(IAL)_Winter_2023_Q8

Solution

To find the integral of the given trigonometric expression in simplest form, we follow these steps:

1. Expand the integrand The integrand is a squared binomial of the form $(a-b)^2 = a^2 - 2ab + b^2$. Applying this to the expression:

$$\begin{aligned}(2\cos x - \sin x)^2 &= (2\cos x)^2 - 2(2\cos x)(\sin x) + (\sin x)^2 \\ &= 4\cos^2 x - 4\sin x\cos x + \sin^2 x\end{aligned}$$

2. Simplify using trigonometric identities To integrate the squared terms and the product term, we use the following **trigonometric identities**:

- **Double-angle formula** for sine: $\sin(2x) = 2\sin x\cos x \implies 4\sin x\cos x = 2\sin(2x)$
- **Power-reduction formulas**:
  - $\cos^2 x = \frac{1+\cos(2x)}{2}$
  - $\sin^2 x = \frac{1-\cos(2x)}{2}$

Substituting these into the expanded expression:

$$\begin{aligned}I &= \int \left(4\left(\frac{1+\cos(2x)}{2}\right) - 2\sin(2x) + \frac{1-\cos(2x)}{2}\right)dx \\ &= \int \left(2 + 2\cos(2x) - 2\sin(2x) + \frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx \\ &= \int \left(\frac{5}{2} + \frac{3}{2}\cos(2x) - 2\sin(2x)\right)dx\end{aligned}$$

3. Perform the integration Now, we integrate each term with respect to x:

$$\int \frac{5}{2}dx = \frac{5}{2}x$$

$$\int \frac{3}{2}\cos(2x)dx = \frac{3}{2}\cdot\frac{\sin(2x)}{2} = \frac{3}{4}\sin(2x)$$

$$\int -2\sin(2x)dx = -2\cdot\left(-\frac{\cos(2x)}{2}\right) = \cos(2x)$$

Combining these results and adding the **constant of integration** C:

$$I = \frac{5}{2}x + \frac{3}{4}\sin(2x) + \cos(2x) + C$$

$$\boxed{\frac{5}{2}x + \frac{3}{4}\sin(2x) + \cos(2x) + C}$$

WMA11_P3(IAL)_Winter_2023_Q9

Solution

1. Differentiation of the Curve Equation

The equation of the curve C is given by:

$$y = \sqrt{3 + 4e^{x^2}} = (3 + 4e^{x^2})^{1/2}$$

To find $\frac{dy}{dx}$, we apply the **chain rule**:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot \frac{d}{dx}(3 + 4e^{x^2}) \\ &= \frac{1}{2}(3 + 4e^{x^2})^{-1/2} \cdot (4e^{x^2} \cdot 2x) \\ &= \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}\end{aligned}$$

Thus, the derivative in its simplest form is:

$$\boxed{\frac{dy}{dx} = \frac{4xe^{x^2}}{\sqrt{3 + 4e^{x^2}}}}$$

2. Tangent Passing Through the Origin

Let the point P have coordinates (α, y_P), where $y_P = \sqrt{3 + 4e^{\alpha^2}}$. The gradient of the **tangent line** at P is:

$$m = \frac{dy}{dx}|_{x=\alpha} = \frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}}$$

Since the tangent passes through the origin $(0, 0)$, its gradient is also given by:

$$m = \frac{y_P - 0}{\alpha - 0} = \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha}$$

Equating the two expressions for the gradient:

$$\begin{aligned}\frac{4\alpha e^{\alpha^2}}{\sqrt{3 + 4e^{\alpha^2}}} &= \frac{\sqrt{3 + 4e^{\alpha^2}}}{\alpha} \\ 4\alpha^2 e^{\alpha^2} &= (\sqrt{3 + 4e^{\alpha^2}})^2 \\ 4\alpha^2 e^{\alpha^2} &= 3 + 4e^{\alpha^2} \\ 4\alpha^2 e^{\alpha^2} - 4e^{\alpha^2} - 3 &= 0\end{aligned}$$

This shows that $x = \alpha$ is a solution to $4x^2 e^{x^2} - 4e^{x^2} - 3 = 0$.

3. Location of the Root α

Let $f(x) = 4x^2 e^{x^2} - 4e^{x^2} - 3$. We use the **Intermediate Value Theorem** by evaluating the function at the boundaries:

- At $x = 1$:

$$f(1) = 4(1)^2 e^{1^2} - 4e^{1^2} - 3 = 4e - 4e - 3 = -3$$

- At $x = 2$:

$$f(2) = 4(2)^2 e^{2^2} - 4e^{2^2} - 3 = 16e^4 - 4e^4 - 3 = 12e^4 - 3 \approx 652.18$$

Since $f(1) < 0$ and $f(2) > 0$, and $f(x)$ is continuous, there must be a root α such that $1 < \alpha < 2$.

4. Rearrangement for Iteration

Starting from the equation in part (b):

$$4x^2 e^{x^2} - 4e^{x^2} - 3 = 0$$
$$4x^2 e^{x^2} = 4e^{x^2} + 3$$
$$x^2 = \frac{4e^{x^2} + 3}{4e^{x^2}}$$
$$x^2 = 1 + \frac{3}{4e^{x^2}}$$
$$x^2 = 1 + \frac{3}{4}e^{-x^2}$$
$$x^2 = \frac{4 + 3e^{-x^2}}{4}$$
$$x = \sqrt{\frac{4 + 3e^{-x^2}}{4}} = \frac{1}{2}\sqrt{4 + 3e^{-x^2}}$$

5. Numerical Iteration

Using the **fixed-point iteration** formula $x_{n+1} = \frac{1}{2}\sqrt{4 + 3e^{-x_n^2}}$ with $x_1 = 1$:

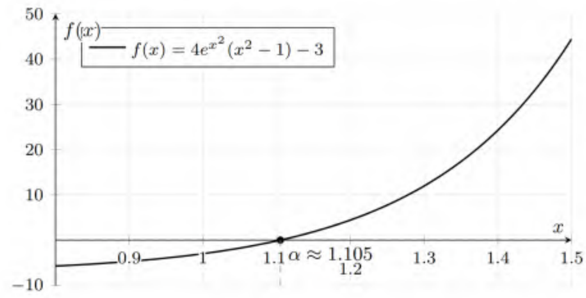
- (i) Finding x_3:

$$x_2 = \frac{1}{2}\sqrt{4 + 3e^{-1^2}} \approx 1.12957...$$
$$x_3 = \frac{1}{2}\sqrt{4 + 3e^{-(1.12957...)^2}} \approx 1.10007...$$

  To 4 decimal places, $x_3 = 1.1001$.
- (ii) Finding α: Continuing the iteration:

$$x_4 \approx 1.1059...$$
$$x_5 \approx 1.1047...$$
$$x_6 \approx 1.1049...$$
$$x_7 \approx 1.1049...$$

  The value converges to 1.1049 to 4 decimal places.



(i) $x_3 = \boxed{1.1001}$

(ii) $\alpha = \boxed{1.1049}$

WMA11_P3(IAL)_Winter_2023_Q10

Solution

The population of fruit flies F at time t (in days) is given by the **logistic model**:

$$F = \frac{350e^{kt}}{9 + e^{kt}}$$

1. Initial population (Part a) To find the number of fruit flies at the start of the study, we evaluate the function at $t = 0$:

$$\begin{aligned} F(0) &= \frac{350e^{k(0)}}{9 + e^{k(0)}} \\ &= \frac{350(1)}{9 + 1} \\ &= \frac{350}{10} \\ &= 35 \quad \text{flies} \end{aligned}$$

$\boxed{35}$

2. Determining the constant k (Part b) Given that at $t = 15$, $F = 200$:

$$\begin{aligned} 200 &= \frac{350e^{15k}}{9 + e^{15k}} \\ 200(9 + e^{15k}) &= 350e^{15k} \\ 1800 + 200e^{15k} &= 350e^{15k} \\ 1800 &= 150e^{15k} \\ e^{15k} &= \frac{1800}{150} \\ e^{15k} &= 12 \end{aligned}$$

Taking the **natural logarithm** of both sides:

$$\begin{aligned} 15k &= \ln(12) \\ k &= \frac{1}{15}\ln(12) \end{aligned}$$

This confirms the required expression for k.

3. Finding the time T for a specific growth rate (Part c) The rate of increase is given by the derivative dF/dt. We apply the **quotient rule**:

$$\begin{aligned} \frac{dF}{dt} &= \frac{(350ke^{kt})(9 + e^{kt}) - (350e^{kt})(ke^{kt})}{(9 + e^{kt})^2} \\ &= \frac{3150ke^{kt} + 350ke^{2kt} - 350ke^{2kt}}{(9 + e^{kt})^2} \\ &= \frac{3150ke^{kt}}{(9 + e^{kt})^2} \end{aligned}$$

We are given that at $t = T$, $dF/dt = 10$. Let $x = e^{kT}$:

$$10 = \frac{3150kx}{(9+x)^2}$$
$$(9+x)^2 = 315kx$$
$$81 + 18x + x^2 = 315\left(\frac{\ln(12)}{15}\right)x$$
$$x^2 + 18x + 81 = 21\ln(12)x$$
$$x^2 + (18 - 21\ln(12))x + 81 = 0$$

Using the **quadratic formula** where $a = 1$, $b = 18 - 21\ln(12) \approx -34.188$, and $c = 81$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x_1 \approx 31.623$$
$$x_2 \approx 2.561$$

Since $x = e^{kT}$, we solve for $T = \frac{\ln(x)}{k} = \frac{15\ln(x)}{\ln(12)}$:

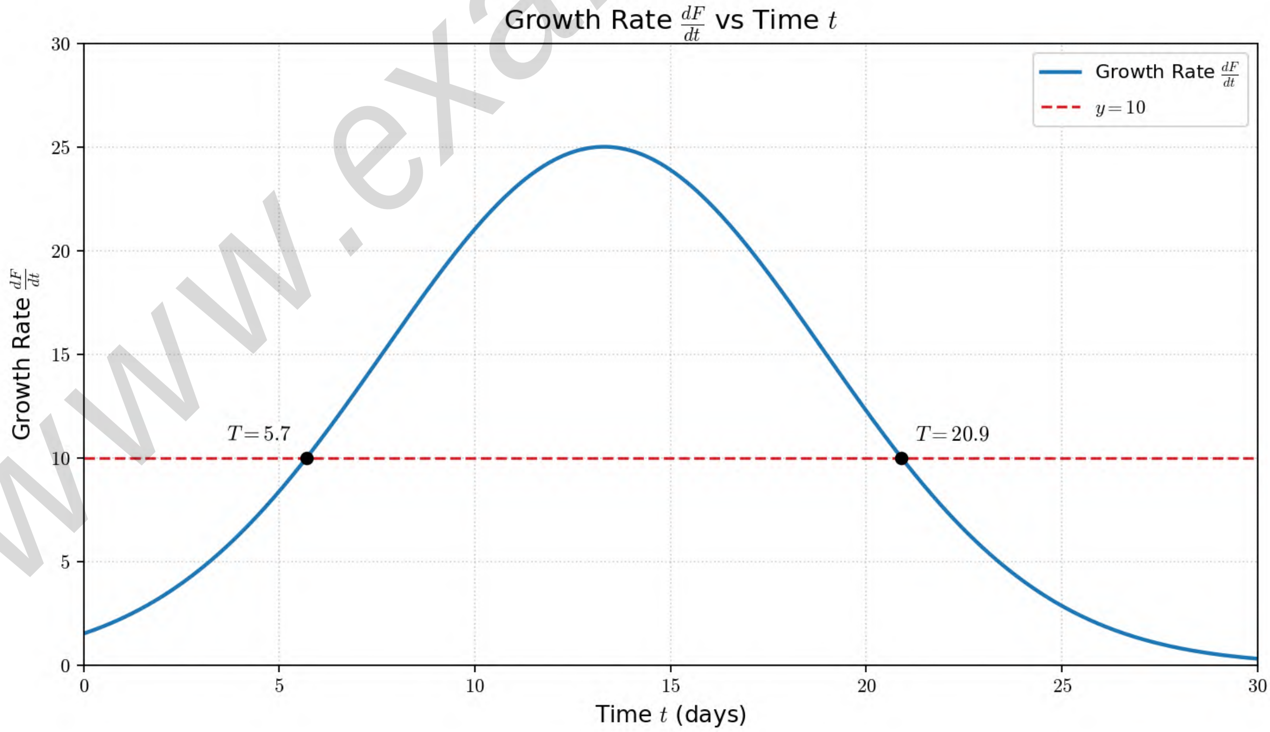
- For $x_1 \approx 31.623$:

$$T_1 = \frac{15\ln(31.623)}{\ln(12)} \approx 20.854$$

- For $x_2 \approx 2.561$:

$$T_2 = \frac{15\ln(2.561)}{\ln(12)} \approx 5.678$$

Rounding to one decimal place, the possible values of T are 5.7 and 20.9.



$$\boxed{T = 5.7, 20.9}$$

WMA11_P3(IAL)_Summer_2024_Q1

Solution

The problem involves analyzing an **absolute value function** $f(x) = 2 \mid x - 5 \mid + 10$, solving an inequality, and applying **function transformations**.

1. Coordinates of the vertex P The general form of an absolute value function is $y = a \mid x - h \mid + k$, where the **vertex** is located at the point (h, k).

- For the function $f(x) = 2 \mid x - 5 \mid + 10$, we identify $h = 5$ and $k = 10$.
- Therefore, the coordinates of the vertex P are $(5, 10)$.

2. Solving the inequality $2 \mid x - 5 \mid + 10 > 6x$ To solve this inequality algebraically, we consider the two cases defined by the absolute value expression $\mid x - 5 \mid$.

- **Case 1: $x \geq 5$** In this region, $\mid x - 5 \mid = x - 5$. The inequality becomes:

$$\begin{aligned} 2(x-5) + 10 &> 6x \\ 2x - 10 + 10 &> 6x \\ 2x &> 6x \\ 0 &> 4x \\ x &< 0 \end{aligned}$$

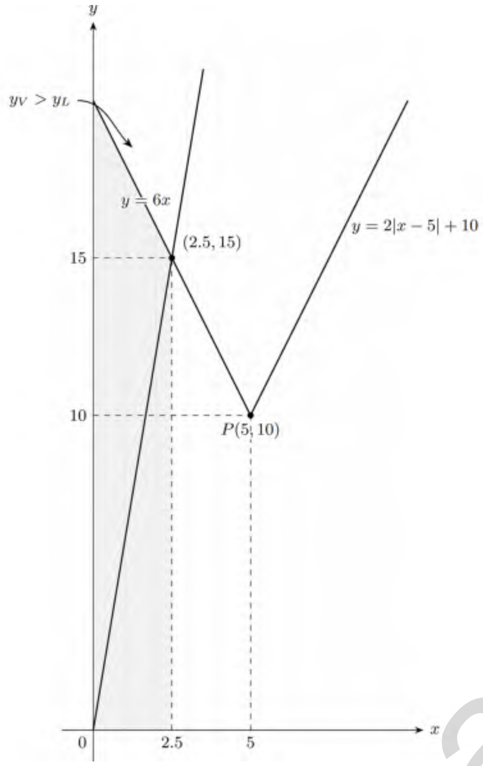
  Since we assumed $x \geq 5$, there are no solutions in this interval because $x < 0$ and $x \geq 5$ are mutually exclusive.

- **Case 2: $x < 5$** In this region, $\mid x - 5 \mid = -(x - 5) = 5 - x$. The inequality becomes:

$$\begin{aligned} 2(5-x) + 10 &> 6x \\ 10 - 2x + 10 &> 6x \\ 20 - 2x &> 6x \\ 20 &> 8x \\ x &< \frac{20}{8} \\ x &< 2.5 \end{aligned}$$

  Since we assumed $x < 5$, the condition $x < 2.5$ is entirely within the valid domain for this case.

Combining the results, the solution to the inequality is $x < 2.5$.



3. Transformation of the vertex P The graph $y = f(x)$ is transformed into $y = 3f(x-2)$. We apply the transformations to the coordinates of $P(5, 10)$ step-by-step:

- **Horizontal Translation:** The term $f(x-2)$ represents a **translation** of 2 units to the right.

$$x_{\text{new}} = 5 + 2 = 7$$

- **Vertical Stretch:** The factor of 3 in $3f(\ldots)$ represents a **vertical stretch** by a scale factor of 3 parallel to the y-axis.

$$y_{\text{new}} = 10 \times 3 = 30$$

The new coordinates of the mapped point are $(7, 30)$.

Final Answers: (a) $P = \boxed{(5, 10)}$ (b) $\boxed{x < 2.5}$ (c) $\boxed{(7, 30)}$

WMA11_P3(IAL)_Summer_2024_Q2

Solution

1. Decomposition of the Rational Function

To express $g(x) = \frac{2x^2-5x+8}{x-2}$ in the form $Ax + B + \frac{C}{x-2}$, we perform **polynomial long division** or use **synthetic division** of the numerator by the denominator.

- Dividing $2x^2 - 5x + 8$ by $x - 2$:
  - The first term of the quotient is $\frac{2x^2}{x} = 2x$.
  - Multiplying $(x-2)$ by $2x$ gives $2x^2 - 4x$.
  - Subtracting this from the numerator: $(2x^2 - 5x + 8) - (2x^2 - 4x) = -x + 8$.
  - The next term of the quotient is $\frac{-x}{x} = -1$.
  - Multiplying $(x-2)$ by -1 gives $-x + 2$.
  - Subtracting this from the remaining part: $(-x+8) - (-x+2) = 6$.

Thus, the quotient is $2x - 1$ and the remainder is 6. We can write:

$$\begin{aligned} g(x) &= \frac{(x-2)(2x-1)+6}{x-2} \\ &= 2x - 1 + \frac{6}{x-2} \end{aligned}$$

Comparing this to the required form $Ax + B + \frac{C}{x-2}$, we identify the integers:

$$A = 2, \quad B = -1, \quad C = 6$$

2. Evaluation of the Definite Integral

We now use the decomposed form of $g(x)$ to evaluate the **definite integral** from $x = 4$ to $x = 8$:

$$I = \int_4^8 \left(2x - 1 + \frac{6}{x-2}\right) dx$$

- Applying the **linearity of integration** and the **power rule**:

$$\begin{aligned} I &= \left[x^2 - x + 6\ln|\ x-2\ |\right]_4^8 \\ &= (8^2 - 8 + 6\ln|\ 8-2\ |) - (4^2 - 4 + 6\ln|\ 4-2\ |) \\ &= (64 - 8 + 6\ln 6) - (16 - 4 + 6\ln 2) \\ &= (56 + 6\ln 6) - (12 + 6\ln 2) \end{aligned}$$

- Using **logarithm laws**, specifically $\ln a - \ln b = \ln(\frac{a}{b})$:

$$\begin{aligned} I &= 56 - 12 + 6(\ln 6 - \ln 2) \\ &= 44 + 6\ln\left(\frac{6}{2}\right) \\ &= 44 + 6\ln 3 \end{aligned}$$

Comparing this result to the form $\alpha + \beta \ln 3$, we find:

$$\alpha = 44, \quad \beta = 6$$

Final Answer: (a) $g(x) = 2x - 1 + \frac{6}{x-2}$ where $\boxed{A = 2, B = -1, C = 6}$ (b) $\int_4^8 g(x)dx = 44 + 6\ln 3$ where $\boxed{\alpha = 44, \beta = 6}$

WMA11_P3(IAL)_Summer_2024_Q3

Solution

1. Linearization of the Power Law Equation

The relationship between x and y is given by the equation:

$$y = \frac{10^6}{x^3}$$

To sketch the graph of $\log_{10} y$ against $\log_{10} x$, we apply the base-10 **logarithm** to both sides of the equation:

$$\begin{aligned}\log_{10} y &= \log_{10}\left(\frac{10^6}{x^3}\right) \\ &= \log_{10}(10^6) - \log_{10}(x^3) \\ &= 6 - 3\log_{10} x\end{aligned}$$

Let $Y = \log_{10} y$ and $X = \log_{10} x$. The equation transforms into a linear form $Y = -3X + 6$, which represents a straight line with a **gradient** of -3 and a vertical intercept of 6.

2. Identifying Intersections with the Axes

- **Vertical Intercept (Y-axis):** Set $X = 0$:

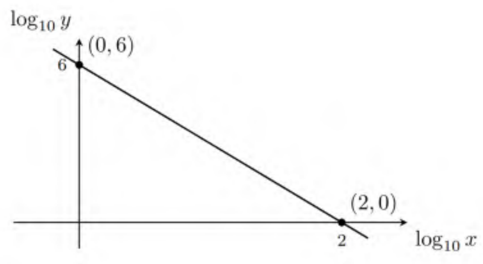
  $$Y = -3(0) + 6 = 6$$

  The point of intersection is $(0, 6)$.

- **Horizontal Intercept (X-axis):** Set $Y = 0$:

  $$\begin{aligned}0 &= -3X + 6 \\ 3X &= 6 \\ X &= 2\end{aligned}$$

  The point of intersection is $(2, 0)$.



3. Determining Constants for the Exponential Model

Figure 2 shows a linear relationship between $\log_3 N$ and t. Let $L = \log_3 N$. The line passes through the points $(-2, 0)$ and $(0, 4)$.

- **Find the equation of the line:** The gradient m is:

$$m = \frac{4-0}{0-(-2)} = \frac{4}{2} = 2$$

Using the intercept $(0, 4)$, the equation is:

$$\log_3 N = 2t + 4$$

- **Convert to the form $N = ab^t$:** Apply the definition of a logarithm (base 3) to isolate N:

$$\begin{aligned} N &= 3^{2t+4} \\ &= 3^4 \cdot 3^{2t} \\ &= 81 \cdot (3^2)^t \\ &= 81 \cdot 9^t \end{aligned}$$

Comparing this to the form $N = ab^t$, we identify the constants:

$$a = 81, \quad b = 9$$

$\boxed{N = 81 \cdot 9^t}$

WMA11_P3(IAL)_Summer_2024_Q4

Solution

1. Simplification of the trigonometric function

To rewrite $f(x) = 8\sin x\cos x + 4\cos^2 x - 3$ in the form $a\sin 2x + b\cos 2x + c$, we apply the **double-angle identities**:

- $\sin 2x = 2\sin x\cos x \implies 8\sin x\cos x = 4(2\sin x\cos x) = 4\sin 2x$
- $\cos 2x = 2\cos^2 x - 1 \implies 2\cos^2 x = \cos 2x + 1 \implies 4\cos^2 x = 2\cos 2x + 2$

Substituting these into the original expression:

$$\begin{aligned} f(x) &= 4\sin 2x + (2\cos 2x + 2) - 3 \\ &= 4\sin 2x + 2\cos 2x - 1 \end{aligned}$$

Comparing this to the form $a\sin 2x + b\cos 2x + c$, we find $a = 4$, $b = 2$, and $c = -1$.

$\boxed{f(x) = 4\sin 2x + 2\cos 2x - 1}$

2. Harmonic form conversion

We now express $4\sin 2x + 2\cos 2x - 1$ in the form $R\sin(2x + \alpha) + c$. Using the **harmonic addition theorem**, for $a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$:

- $R = \sqrt{a^2 + b^2}$
- $\tan\alpha = \frac{b}{a}$

Calculating the constants:

$$\begin{aligned} R &= \sqrt{4^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5} \\ \tan\alpha &= \frac{2}{4} = 0.5 \\ \alpha &= \arctan(0.5) \approx 0.4636476... \end{aligned}$$

Rounding α to 3 significant figures, we get $\alpha \approx 0.464\,\text{rad}$.

$\boxed{f(x) = 2\sqrt{5}\sin(2x + 0.464) - 1}$

3. Maximum values and their locations

- **(i) Maximum value of $f(x)$** The maximum value of the sine function is 1. Therefore, the maximum value of $f(x)$ occurs when $\sin(2x + \alpha) = 1$:

$$\begin{aligned} f_{\max} &= R(1) + c \\ &= 2\sqrt{5} - 1 \end{aligned}$$

$\boxed{2\sqrt{5} - 1}$

- **(ii) Second smallest positive value of x for the maximum** The maximum occurs when the argument of the sine function satisfies:

$$2x + \alpha = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$$

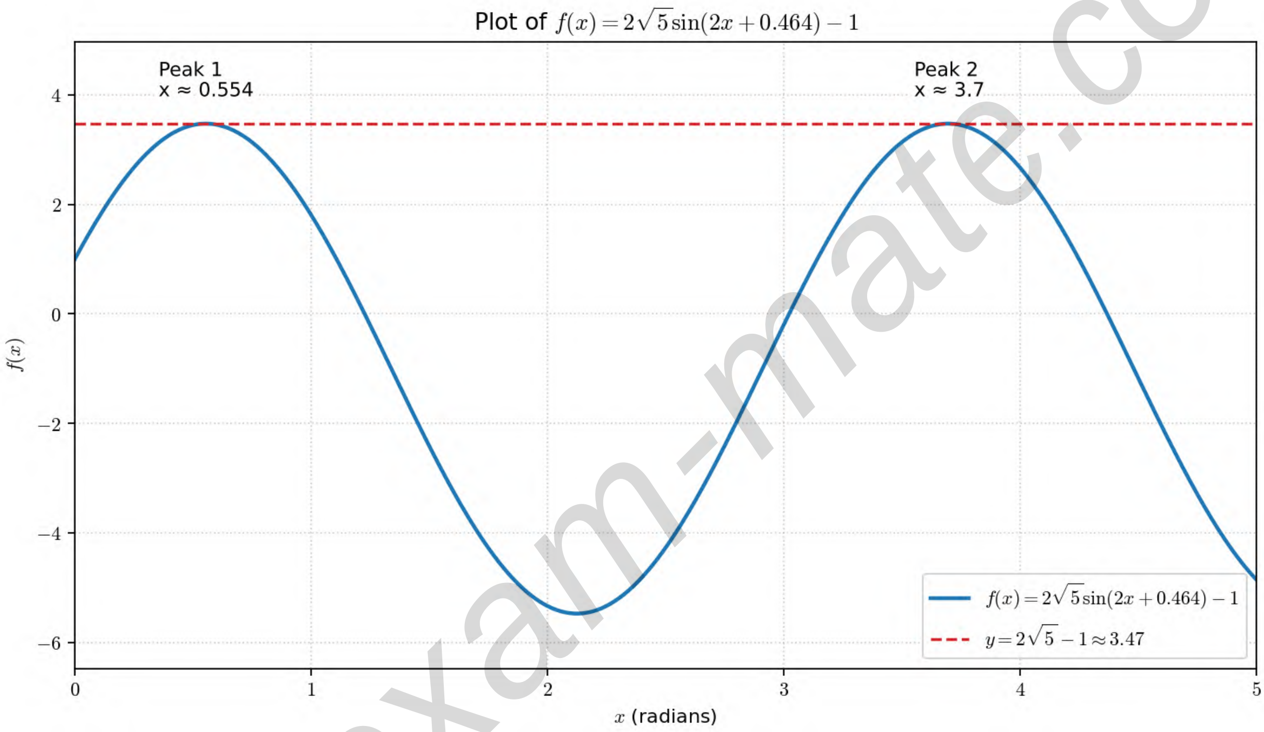
Solving for x:

$$x = \frac{\frac{\pi}{2} - \alpha + 2k\pi}{2}$$

Using the exact value of $\alpha = \arctan(0.5) \approx 0.4636476$:

- For $k = 0$: $x_1 = \frac{\frac{\pi}{2} - 0.4636476}{2} \approx \frac{1.570796 - 0.4636476}{2} \approx 0.55357...$
- For $k = 1$: $x_2 = \frac{\frac{\pi}{2} - 0.4636476 + 2\pi}{2} = x_1 + \pi \approx 0.55357 + 3.14159 \approx 3.69516...$

The second smallest positive value is x_2. Rounding to 3 significant figures: $\boxed{3.70}$



WMA11_P3(IAL)_Summer_2024_Q5

Solution

1. Finding $f^{-1}(22)$

To find the value of the **inverse function** $f^{-1}(22)$, we set the original function $f(x)$ equal to 22 and solve for x:

$$f(x) = 2 + 5\ln x$$
$$22 = 2 + 5\ln x$$
$$20 = 5\ln x$$
$$4 = \ln x$$
$$x = e^4$$

Since $x = e^4 > 0$, it lies within the domain of f. Thus, $f^{-1}(22) = e^4$.

$\boxed{e^4}$

2. Proving g is an increasing function

To prove that $g(x)$ is an **increasing function**, we calculate its first derivative using the **quotient rule**:

$$g(x) = \frac{6x-2}{2x+1}$$
$$g'(x) = \frac{(6)(2x+1)-(6x-2)(2)}{(2x+1)^2}$$
$$= \frac{12x+6-(12x-4)}{(2x+1)^2}$$
$$= \frac{10}{(2x+1)^2}$$

For all x in the domain $x > 1/3$, the denominator $(2x+1)^2$ is always positive. Since the numerator is the positive constant 10, it follows that $g'(x) > 0$ for all $x > 1/3$. Therefore, g is a strictly increasing function.

3. Finding g^{-1}

To find the expression for $g^{-1}(x)$, we let $y = g(x)$ and solve for x in terms of y:

$$y = \frac{6x-2}{2x+1}$$
$$y(2x+1) = 6x-2$$
$$2xy + y = 6x - 2$$
$$y + 2 = 6x - 2xy$$
$$y + 2 = x(6-2y)$$
$$x = \frac{y+2}{6-2y}$$

Swapping the variables to express the inverse function:

$$g^{-1}(x) = \frac{x+2}{6-2x}$$

The domain of g^{-1} is the range of g. Since g is increasing, we check the boundaries:

- As $x \to 1/3^+$, $g(x) \to \frac{6(1/3)-2}{2(1/3)+1} = \frac{0}{5/3} = 0$.
- As $x \to \infty$, $g(x) \to \frac{6}{2} = 3$ (horizontal asymptote). Thus, the domain of g^{-1} is $0 < x < 3$.

$$\boxed{g^{-1}(x) = \frac{x+2}{6-2x}, \quad 0 < x < 3}$$

4. Finding the range of fg

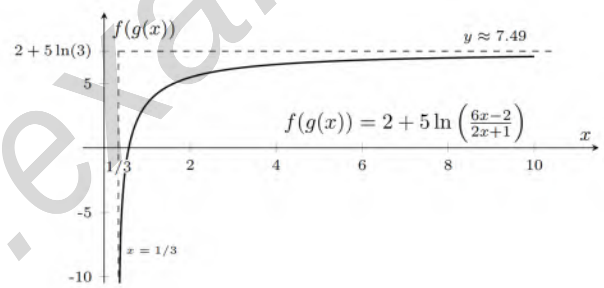
The **composite function** fg is defined as $f(g(x))$.

- First, we determine the range of $g(x)$ for its domain $x > 1/3$. As established in step 3, the range of g is $(0, 3)$.
- Next, we find the range of f when its input is restricted to the interval $(0, 3)$.

$$fg(x) = f(g(x)) = 2 + 5\ln(g(x))$$

  Since $f(x) = 2 + 5\ln x$ is a strictly increasing function for $x > 0$:

- As $g(x) \to 0^+$, $f(g(x)) \to -\infty$.
- As $g(x) \to 3^-$, $f(g(x)) \to 2 + 5\ln 3$.



The range of fg is therefore $(-\infty, 2 + 5\ln 3)$.

$$\boxed{fg(x) < 2 + 5\ln 3}$$

WMA11_P3(IAL)_Summer_2024_Q6

Solution

1. Equation of the normal line

To find the equation of the **normal line** l to the curve $y = \sqrt{4x-7}$ at the point $P(8,5)$, we first determine the gradient of the **tangent** at that point.

- The equation of the curve is:

$$y = (4x-7)^{1/2}$$

- Using the **chain rule**, the derivative is:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}(4x-7)^{-1/2} \cdot 4 \\ &= \frac{2}{\sqrt{4x-7}}\end{aligned}$$

- At the point $P(8,5)$, where $x = 8$:

$$\begin{aligned}m_{\text{tangent}} &= \frac{2}{\sqrt{4(8)-7}} \\ &= \frac{2}{\sqrt{25}} \\ &= \frac{2}{5}\end{aligned}$$

- The gradient of the normal, m_l, is the negative reciprocal of the tangent gradient:

$$m_l = -\frac{1}{m_{\text{tangent}}} = -\frac{5}{2}$$

- Using the point-slope form $y - y_1 = m(x - x_1)$ for $P(8,5)$:

$$\begin{aligned}y - 5 &= -\frac{5}{2}(x-8) \\ 2y - 10 &= -5x + 40 \\ 5x + 2y - 50 &= 0\end{aligned}$$

  This confirms the given equation for line l.

2. Area of the shaded region R

The region R is bounded by the curve, the x-axis, and the line l. To find the total area, we split the region into two parts at the x-coordinate of point P ($x = 8$).

- **Part 1: Area under the curve from the x-intercept to $x = 8$** The curve $y = \sqrt{4x-7}$ intersects the x-axis when $y = 0$:

$$4x - 7 = 0 \implies x = 1.75$$

  The area A_1 is given by the **definite integral**:

$$\begin{aligned} A_1 &= \int_{1.75}^{8} (4x-7)^{1/2} dx \\ &= \left[\frac{(4x-7)^{3/2}}{4 \cdot \frac{3}{2}} \right]_{1.75}^{8} \\ &= \left[\frac{1}{6}(4x-7)^{3/2} \right]_{1.75}^{8} \\ &= \frac{1}{6}(4(8)-7)^{3/2} - \frac{1}{6}(4(1.75)-7)^{3/2} \\ &= \frac{1}{6}(25)^{3/2} - 0 \\ &= \frac{125}{6} \end{aligned}$$

- **Part 2: Area under the line l from $x = 8$ to its x-intercept** The line $5x + 2y - 50 = 0$ intersects the x-axis when $y = 0$:

  $$5x = 50 \implies x = 10$$

  This region is a right-angled triangle with base $b = 10 - 8 = 2$ and height $h = 5$.

  $$\begin{aligned} A_2 &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 2 \times 5 \\ &= 5 \end{aligned}$$

- **Total Area R**

  $$\begin{aligned} \text{Total Area} &= A_1 + A_2 \\ &= \frac{125}{6} + 5 \\ &= \frac{125+30}{6} \\ &= \frac{155}{6} \end{aligned}$$

[Visualization]

The exact area of region R is:

$$\boxed{\frac{155}{6}}$$

WMA11_P3(IAL)_Summer_2024_Q7

Solution

1. Derivation of the tangent identity

To show that $\tan x = -2 - \sqrt{3}$ from the given equation $\sqrt{2}\sin(x + 45^\circ) = \cos(x - 60^\circ)$, we apply the **addition and subtraction formulas** for sine and cosine:

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Expanding both sides of the equation:

$$\sqrt{2}(\sin x \cos 45^\circ + \cos x \sin 45^\circ) = \cos x \cos 60^\circ + \sin x \sin 60^\circ$$

Using the exact values $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 60^\circ = \frac{1}{2}$, and $\sin 60^\circ = \frac{\sqrt{3}}{2}$:

$$\sqrt{2}\left(\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}}\right) = \cos x \cdot \frac{1}{2} + \sin x \cdot \frac{\sqrt{3}}{2}$$
$$\sin x + \cos x = \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$$

Multiply the entire equation by 2 to clear the fractions:

$$2\sin x + 2\cos x = \cos x + \sqrt{3}\sin x$$
$$(2 - \sqrt{3})\sin x = -\cos x$$

To find $\tan x$, we divide by $\cos x$ (assuming $\cos x \neq 0$) and by $(2 - \sqrt{3})$:

$$\frac{\sin x}{\cos x} = \frac{-1}{2 - \sqrt{3}}$$
$$\tan x = \frac{-1}{2 - \sqrt{3}}$$

Rationalize the denominator by multiplying the numerator and denominator by the **conjugate** $(2 + \sqrt{3})$:

$$\tan x = \frac{-1(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})}$$
$$= \frac{-2 - \sqrt{3}}{4 - 3}$$
$$= -2 - \sqrt{3}$$

2. Solving the trigonometric equation

We are asked to solve $\sqrt{2}\sin(2\theta) = \cos(2\theta - 105^\circ)$ for $0^\circ \leq \theta < 180^\circ$. Let $x = 2\theta - 45^\circ$. Then $2\theta = x + 45^\circ$ and $2\theta - 105^\circ = (x + 45^\circ) - 105^\circ = x - 60^\circ$. Substituting these into the equation:

$$\sqrt{2}\sin(x + 45^\circ) = \cos(x - 60^\circ)$$

This matches the form derived in part (a). Therefore, we have:

$$\tan x = -2 - \sqrt{3}$$

- **Step 1: Find the principal value for x** Using the **inverse tangent function**, we find the angle whose tangent is $-(2+\sqrt{3})$. Since $\tan 75^\circ = 2+\sqrt{3}$, it follows that:

$$x = \arctan(-2-\sqrt{3}) = -75^\circ$$

- **Step 2: Determine the range for x** Given $0^\circ \le \theta < 180^\circ$, the range for 2θ is $0^\circ \le 2\theta < 360^\circ$. The range for $x = 2\theta - 45^\circ$ is:

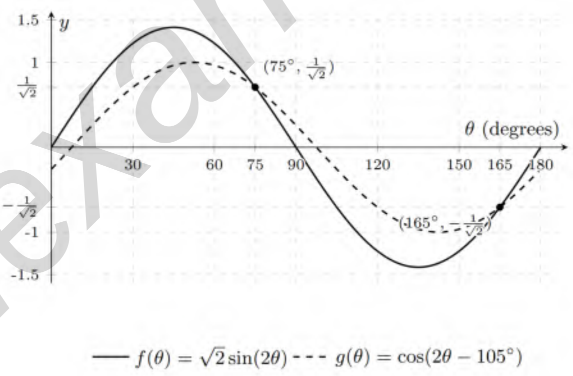
$$-45^\circ \le x < 315^\circ$$

- **Step 3: Solve for x within the range** The general solution for $\tan x = C$ is $x = \alpha + 180^\circ k$. For $k=0$: $x = -75^\circ$ (Outside range) For $k=1$: $x = -75^\circ + 180^\circ = 105^\circ$ For $k=2$: $x = -75^\circ + 360^\circ = 285^\circ$

- **Step 4: Solve for θ** Using $2\theta = x + 45^\circ$:
  - For $x = 105^\circ$:

    $$2\theta = 105^\circ + 45^\circ = 150^\circ \implies \theta = 75^\circ$$
  - For $x = 285^\circ$:

    $$2\theta = 285^\circ + 45^\circ = 330^\circ \implies \theta = 165^\circ$$



$\boxed{\theta = 75^\circ, 165^\circ}$

WMA11_P3(IAL)_Summer_2024_Q8

Solution

The path of the golf ball is modeled by the equation:

$$h = 1.5x - 0.5xe^{0.02x}, \quad 0 \leq x \leq d$$

where h is the vertical height in metres and x is the horizontal distance in metres.

1. Finding the value of d The value d represents the horizontal distance when the ball first hits the ground. At this point, the height $h = 0$.

- Setting the equation to zero:

$$1.5x - 0.5xe^{0.02x} = 0$$

- Factoring out $0.5x$:

$$0.5x(3 - e^{0.02x}) = 0$$

- Since $d > 0$, we ignore the solution $x = 0$ (the starting point) and solve for the term in the parentheses:

$$\begin{aligned} 3 - e^{0.02x} &= 0 \\ e^{0.02x} &= 3 \\ 0.02x &= \ln(3) \\ x &= \frac{\ln(3)}{0.02} \\ x &= 50\ln(3) \end{aligned}$$

- Calculating the numerical value:

$$x \approx 54.9306...$$

  Rounding to 2 decimal places, we find:

$$\boxed{d = 54.93}$$

2. Maximum value of h To find the maximum height, we determine the stationary point by differentiating h with respect to x using the **Product Rule** for the second term.

- Let $h = 1.5x - 0.5xe^{0.02x}$.
- The derivative is:

$$\begin{aligned} \frac{dh}{dx} &= 1.5 - [0.5(1)e^{0.02x} + 0.5x(0.02e^{0.02x})] \\ &= 1.5 - 0.5e^{0.02x} - 0.01xe^{0.02x} \\ &= 1.5 - e^{0.02x}(0.5 + 0.01x) \end{aligned}$$

- Setting $\frac{dh}{dx} = 0$ for the maximum:

$$1.5 = e^{0.02x}(0.5 + 0.01x)$$
$$e^{0.02x} = \frac{1.5}{0.5 + 0.01x}$$

- Multiply the numerator and denominator by 100 to simplify:

$$e^{0.02x} = \frac{150}{50 + x}$$

- Taking the natural logarithm of both sides:

$$0.02x = \ln\left(\frac{150}{x + 50}\right)$$
$$x = \frac{1}{0.02}\ln\left(\frac{150}{x + 50}\right)$$
$$x = 50\ln\left(\frac{150}{x + 50}\right)$$

This confirms the required **iteration formula**.

3. Iteration results Using the formula $x_{n+1} = 50\ln\left(\frac{150}{x_n+50}\right)$ with an initial value $x_1 = 30$:

- **(i) Value of x_2:**

$$x_2 = 50\ln\left(\frac{150}{30 + 50}\right)$$
$$= 50\ln\left(\frac{150}{80}\right)$$
$$= 50\ln(1.875)$$
$$\approx 31.429...$$

Rounding to 2 decimal places:

$$\boxed{x_2 = 31.43}$$

- **(ii) Repeated iteration for maximum height distance:** Continuing the process to find the **convergence** point:

$$x_3 = 50\ln\left(\frac{150}{31.429... + 50}\right) \approx 30.54...$$
$$x_4 = 50\ln\left(\frac{150}{30.54... + 50}\right) \approx 31.09...$$
$$x_5 = 50\ln\left(\frac{150}{31.09... + 50}\right) \approx 30.75...$$
$$\vdots$$
$$x_\infty \approx 30.88$$

The horizontal distance travelled before the ball reaches its maximum height is:

$$\boxed{30.88\,\text{m}}$$

WMA11_P3(IAL)_Summer_2024_Q9

Solution

The curve is defined by the equation $x = 4\sin^2 y - 1$ for the interval $0 \le y \le \frac{\pi}{2}$.

1. Verification of the constant k The point $P(k, \frac{\pi}{3})$ lies on the curve. Substituting $y = \frac{\pi}{3}$ into the equation:

$$\begin{aligned} k &= 4\sin^2\left(\frac{\pi}{3}\right) - 1 \\ &= 4\left(\frac{\sqrt{3}}{2}\right)^2 - 1 \\ &= 4\left(\frac{3}{4}\right) - 1 \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

Thus, $k = 2$ is verified.

2. Differentiation

- **(i) Finding dx/dy** Applying the **chain rule** to $x = 4(\sin y)^2 - 1$:

$$\begin{aligned} \frac{dx}{dy} &= 4 \cdot 2\sin y \cos y \\ &= 8\sin y \cos y \end{aligned}$$

  Using the **double angle formula** $\sin(2y) = 2\sin y \cos y$, this can also be written as $4\sin(2y)$.

- **(ii) Showing the expression for dy/dx** We use the identity $\frac{dy}{dx} = \frac{1}{dx/dy}$. From the original equation:

$$\begin{aligned} x + 1 = 4\sin^2 y &\implies \sin y = \frac{\sqrt{x+1}}{2} \\ 4 - (x+1) = 4 - 4\sin^2 y &= 4(1 - \sin^2 y) = 4\cos^2 y \\ 3 - x = 4\cos^2 y &\implies \cos y = \frac{\sqrt{3-x}}{2} \end{aligned}$$

  Substituting these into the expression for dx/dy:

$$\begin{aligned} \frac{dx}{dy} &= 8\left(\frac{\sqrt{x+1}}{2}\right)\left(\frac{\sqrt{3-x}}{2}\right) \\ &= 2\sqrt{x+1}\sqrt{3-x} \end{aligned}$$

  Therefore:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}\sqrt{3-x}}$$

3. Area of triangle OPN The **normal** to the curve at $P(2, \frac{\pi}{3})$ is perpendicular to the **tangent**.

- First, find the gradient of the tangent m_t at P:

$$m_t = \frac{dy}{dx}|_{x=2} = \frac{1}{2\sqrt{2+1}\sqrt{3-2}}$$
$$= \frac{1}{2\sqrt{3}}$$

- The gradient of the normal m_n is the negative reciprocal:

$$m_n = -\frac{1}{m_t} = -2\sqrt{3}$$

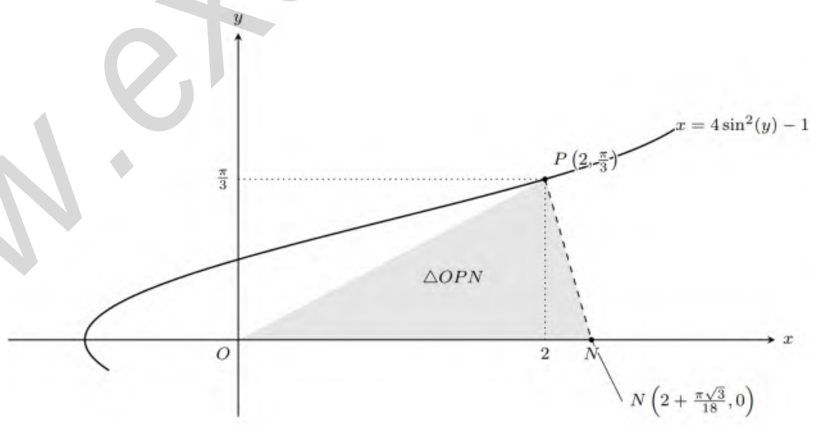
- The equation of the normal line at $P(2, \frac{\pi}{3})$ is:

$$y - \frac{\pi}{3} = -2\sqrt{3}(x-2)$$

- Point N is the x-intercept of this normal (where $y = 0$):

$$-\frac{\pi}{3} = -2\sqrt{3}(x_N - 2)$$
$$x_N - 2 = \frac{\pi}{6\sqrt{3}}$$
$$x_N = 2 + \frac{\pi\sqrt{3}}{18}$$

- The triangle OPN has vertices $O(0,0)$, $P(2, \frac{\pi}{3})$, and $N(2 + \frac{\pi\sqrt{3}}{18}, 0)$.



Using the formula for the area of a triangle with a base on the x-axis:

$$\begin{aligned}\text{Area} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times x_N \times y_P \\ &= \frac{1}{2}\left(2 + \frac{\pi\sqrt{3}}{18}\right)\left(\frac{\pi}{3}\right) \\ &= \frac{\pi}{3} + \frac{\pi^2\sqrt{3}}{108}\end{aligned}$$

Comparing this to the form $a\pi + b\pi^2$, we have $a = \frac{1}{3}$ and $b = \frac{\sqrt{3}}{108}$.

$$\boxed{\frac{1}{3}\pi + \frac{\sqrt{3}}{108}\pi^2}$$

WMA11_P3(IAL)_Winter_2024_Q1

Solution

To find the image of the point $P(-4, -3)$ under various **function transformations**, we utilize the property that if a point (x_0, y_0) lies on the curve $y = f(x)$, then $f(x_0) = y_0$. For the given point P, we have $f(-4) = -3$.

1. Transformation (a): $y = f(2x)$

- This transformation represents a **horizontal compression** by a factor of $\frac{1}{2}$ toward the y-axis.
- To find the new x-coordinate, we set the argument of the function equal to the original x-value:

$$\begin{aligned} 2x &= -4 \\ x &= -2 \end{aligned}$$

- The y-coordinate remains the same because the transformation is applied only to the input x:

$$y = f(2(-2)) = f(-4) = -3$$

- The mapped point is $(-2, -3)$.

2. Transformation (b): $y = 3f(x-1)$

- This transformation involves two steps: a **horizontal translation** to the right by 1 unit and a **vertical stretch** by a factor of 3.
- For the horizontal component:

$$\begin{aligned} x - 1 &= -4 \\ x &= -3 \end{aligned}$$

- For the vertical component:

$$\begin{aligned} y &= 3f(-3-1) \\ &= 3f(-4) \\ &= 3(-3) \\ &= -9 \end{aligned}$$

- The mapped point is $(-3, -9)$.

3. Transformation (c): $y = |\ f(x)\ |$

- This transformation is a **reflection** of the parts of the graph below the x-axis across the x-axis.
- The x-coordinate remains unchanged: $x = -4$.
- The y-coordinate is the absolute value of the original y-value:

$$\begin{aligned} y &= |\ f(-4)\ | \\ &= |-3\ | \\ &= 3 \end{aligned}$$

- The mapped point is $(-4, 3)$.

Final Results: (a) $\boxed{(-2, -3)}$ (b) $\boxed{(-3, -9)}$ (c) $\boxed{(-4, 3)}$

WMA11_P3(IAL)_Winter_2024_Q2

Solution

1. Verification of the root in the interval $[2, 3]$

To show that the equation $f(x) = 0$ has a root α in the interval $[2, 3]$, we apply the **Intermediate Value Theorem**. Since $f(x) = x^4 - 5x^2 + 4x - 7$ is a polynomial, it is continuous for all $x \in \mathbb{R}$.

- Evaluate $f(x)$ at the boundaries:

$$\begin{aligned} f(2) &= (2)^4 - 5(2)^2 + 4(2) - 7 \\ &= 16 - 20 + 8 - 7 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(3) &= (3)^4 - 5(3)^2 + 4(3) - 7 \\ &= 81 - 45 + 12 - 7 \\ &= 41 \end{aligned}$$

Since $f(2) = -3 < 0$ and $f(3) = 41 > 0$, there is a change of sign in the interval $[2, 3]$. Because the function is continuous, there must exist at least one value $\alpha \in (2, 3)$ such that $f(\alpha) = 0$.

2. Rearrangement of the equation $f(x) = 0$

We aim to rewrite $x^4 - 5x^2 + 4x - 7 = 0$ in the form $x = \sqrt[3]{\frac{5x^2-4x+7}{x}}$.

- Start with the equation:

$$x^4 - 5x^2 + 4x - 7 = 0$$

- Isolate the x^4 term:

$$x^4 = 5x^2 - 4x + 7$$

- Divide both sides by x (assuming $x \neq 0$, which is valid for the interval $[2, 3]$):

$$x^3 = \frac{5x^2 - 4x + 7}{x}$$

- Take the cube root of both sides:

$$\boxed{x = \sqrt[3]{\frac{5x^2 - 4x + 7}{x}}}$$

3. Iterative calculation

Using the **fixed-point iteration** formula $x_{n+1} = \sqrt[3]{\frac{5x_n^2-4x_n+7}{x_n}}$ with the initial value $x_1 = 2$:

- **(i) Find x_2 to 4 decimal places:**

$$\begin{aligned} x_2 &= \sqrt[3]{\frac{5(2)^2 - 4(2) + 7}{2}} \\ &= \sqrt[3]{\frac{20 - 8 + 7}{2}} \\ &= \sqrt[3]{\frac{19}{2}} \\ &= \sqrt[3]{9.5} \\ &\approx 2.117911... \end{aligned}$$

Rounding to 4 decimal places:

$$\boxed{x_2 = 2.1179}$$

- **(ii) Find the value of α to 4 decimal places:** Continuing the iteration process:

$$x_3 = \sqrt[3]{\frac{5(2.117911)^2 - 4(2.117911) + 7}{2.117911}} \approx 2.0800$$

$$x_4 = \sqrt[3]{\frac{5(2.080036)^2 - 4(2.080036) + 7}{2.080036}} \approx 2.0914$$

$$x_5 = \sqrt[3]{\frac{5(2.091436)^2 - 4(2.091436) + 7}{2.091436}} \approx 2.0879$$

$$x_6 = \sqrt[3]{\frac{5(2.087913)^2 - 4(2.087913) + 7}{2.087913}} \approx 2.0890$$

$$x_7 = \sqrt[3]{\frac{5(2.088995)^2 - 4(2.088995) + 7}{2.088995}} \approx 2.0887$$

$$x_8 = \sqrt[3]{\frac{5(2.088662)^2 - 4(2.088662) + 7}{2.088662}} \approx 2.0888$$

$$x_9 = \sqrt[3]{\frac{5(2.088764)^2 - 4(2.088764) + 7}{2.088764}} \approx 2.0887$$

The values converge to 2.0888 when rounded to 4 decimal places. To verify, we check the sign of $f(x)$ at the bounds of the rounding interval $[2.08875, 2.08885]$: $f(2.08875) \approx -0.0006$ $f(2.08885) \approx 0.0008$ Since there is a sign change, the root α is correctly identified.

$$\boxed{\alpha = 2.1565}$$

WMA11_P3(IAL)_Winter_2024_Q3

Solution

1. Converting the logarithmic model to exponential form

The relationship between the total amount raised D and the time t is given by the **logarithmic equation**:

$$\log_{10} D = 1.04 + 0.38t$$

To express this in the form $D = ab^t$, we apply the inverse operation of the base-10 logarithm, which is the **exponential function** 10^x:

$$\begin{aligned} D &= 10^{1.04+0.38t} \\ &= 10^{1.04} \cdot (10^{0.38})^t \end{aligned}$$

Comparing this to the form $D = ab^t$, we identify the constants a and b:

- $a = 10^{1.04}$
- $b = 10^{0.38}$

Calculating these values to 4 significant figures:

- $a = 10.96478... \approx 10.96$
- $b = 2.39883... \approx 2.399$

Thus, the equation is:

$$\boxed{D = 10.96 \times 2.399^t}$$

2. Finding the value of T when $D = 45\,000$

We substitute $D = 45\,000$ into the original logarithmic equation to solve for $t = T$:

$$\begin{aligned} \log_{10}(45\,000) &= 1.04 + 0.38T \\ 4.65321... &= 1.04 + 0.38T \\ 0.38T &= 4.65321... - 1.04 \\ 0.38T &= 3.61321... \\ T &= \frac{3.61321...}{0.38} \\ T &= 9.50845... \end{aligned}$$

Rounding to 3 significant figures:

$$\boxed{T = 9.51}$$

3. Determining if the charity achieves its aim

The charity aims to raise £$350\,000$ within $t = 12$ months. We calculate the predicted amount D at $t = 12$:

$$\log_{10} D = 1.04 + 0.38(12)$$
$$\log_{10} D = 1.04 + 4.56$$
$$\log_{10} D = 5.60$$
$$D = 10^{5.60}$$
$$D = 398\,107.17\ldots$$

Comparing the predicted amount to the target:

- Predicted amount: £398 107.17
- Target amount: £350 000.00

Since $398\,107.17 > 350\,000$, the model predicts that the charity will achieve its aim.

The charity will achieve its aim.

WMA11_P3(IAL)_Winter_2024_Q4

Solution

1. Simplification of $f(x)$

To show that $f(x) = \frac{2x}{3x-5}$, we first factorize the components of the expression.

- The numerator of the first term is a **difference of squares**:

$$2x^2 - 32 = 2(x^2 - 16) = 2(x-4)(x+4)$$

- The denominator of the first term is a quadratic expression:

$$3x^2 + 7x - 20 = (3x-5)(x+4)$$

- Substituting these back into $f(x)$:

$$\begin{aligned} f(x) &= \frac{2(x-4)(x+4)}{(3x-5)(x+4)} + \frac{8}{3x-5} \\ &= \frac{2(x-4)}{3x-5} + \frac{8}{3x-5} \\ &= \frac{2x-8+8}{3x-5} \\ &= \frac{2x}{3x-5} \end{aligned}$$

  Since $x > 2$, the term $(x+4) \neq 0$, justifying the cancellation.

2. Analysis of the monotonicity of $f(x)$

To determine if f is a decreasing function, we compute its first derivative using the **Quotient Rule**:

- Let $u = 2x$ and $v = 3x - 5$. Then $u' = 2$ and $v' = 3$.

$$\begin{aligned} f'(x) &= \frac{u'v - uv'}{v^2} \\ &= \frac{2(3x-5) - (2x)(3)}{(3x-5)^2} \\ &= \frac{6x - 10 - 6x}{(3x-5)^2} \\ &= \frac{-10}{(3x-5)^2} \end{aligned}$$

  For all x in the domain ($x > 2$), the denominator $(3x-5)^2$ is always positive. Since the numerator is -10, it follows that $f'(x) < 0$ for all $x > 2$. Therefore, f is a **decreasing function**.

3. Finding the inverse function g^{-1}

The function is defined as $g(x) = 3 + 2\ln x$ for $x \geq 1$.

- Set $y = 3 + 2\ln x$ and solve for x:

$$y - 3 = 2\ln x$$
$$\frac{y-3}{2} = \ln x$$
$$x = e^{\frac{y-3}{2}}$$

- Swapping variables, we obtain the **inverse function**:

$$\boxed{g^{-1}(x) = e^{\frac{x-3}{2}}}$$

The domain of g^{-1} is the range of g. Since $x \geq 1$, $\ln x \geq 0$, so $g(x) \geq 3$. Thus, the domain of g^{-1} is $x \geq 3$.

4. Solving for a in $gf(a) = 5$

We are given the **composite function** equation $g(f(a)) = 5$.

- First, apply the inverse function g^{-1} to both sides:

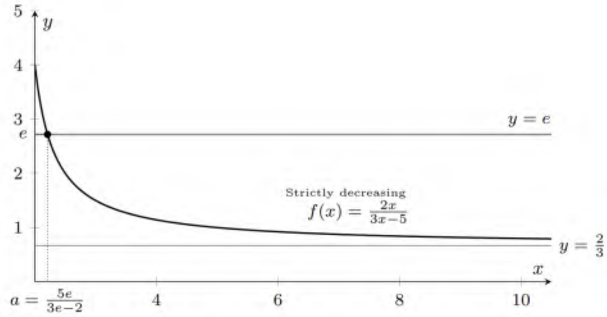
$$f(a) = g^{-1}(5)$$

- Using the expression for g^{-1} found in part (c):

$$f(a) = e^{\frac{5-3}{2}} = e^1 = e$$

- Substitute the simplified form of $f(a)$:

$$\frac{2a}{3a-5} = e$$
$$2a = e(3a-5)$$
$$2a = 3ea - 5e$$
$$5e = 3ea - 2a$$
$$5e = a(3e-2)$$
$$a = \frac{5e}{3e-2}$$


y
y = e
e
Strictly decreasing
$f(x) = \frac{2x}{3x-5}$
$y = \frac{2}{3}$
x
$a = \frac{5e}{3e-2}$
5
4
3
2
1
4
6
8
10

$$a = \frac{5e}{3e-2}$$

WMA11_P3(IAL)_Winter_2024_Q5

Solution

The temperature T in $^\circ$ C at time t minutes is modeled by **Newton's Law of Cooling** in the form:

$$T = 10 + Ae^{-Bt}$$

1. Finding the value of A - The temperature at the instant the heat source was switched off ($t = 0$) was 18° C. - Substituting $t = 0$ and $T = 18$ into the equation:

$$\begin{aligned} 18 &= 10 + Ae^{-B(0)} \\ 18 &= 10 + A(1) \\ A &= 18 - 10 \\ A &= 8 \end{aligned}$$

- Thus, $\boxed{A = 8}$.

2. Finding the value of B - At $t = 45\,\text{min}$, the temperature $T = 16^\circ$ C. Using $A = 8$:

$$\begin{aligned} 16 &= 10 + 8e^{-45B} \\ 6 &= 8e^{-45B} \\ \frac{6}{8} &= e^{-45B} \\ 0.75 &= e^{-45B} \end{aligned}$$

- Taking the **natural logarithm** ($\ln$) of both sides:

$$\begin{aligned} \ln(0.75) &= -45B \\ B &= \frac{\ln(0.75)}{-45} \\ B &\approx 0.0063927\ldots \end{aligned}$$

- To 3 significant figures, $\boxed{B = 0.00639}$.

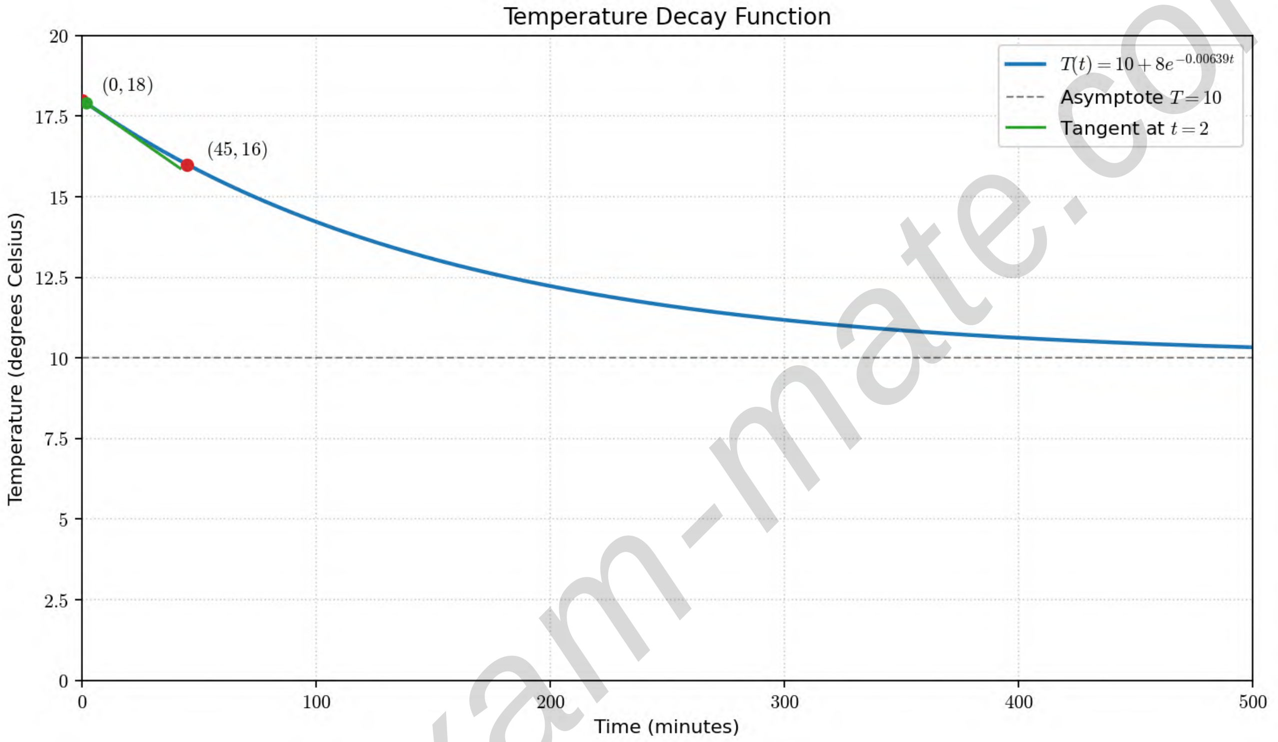
3. Finding the rate of change of temperature at $t = 2$ - The **rate of change** is given by the derivative dT/dt:

$$\begin{aligned} T &= 10 + 8e^{-Bt} \\ \frac{dT}{dt} &= -8Be^{-Bt} \end{aligned}$$

- Substituting $t = 2$ and the unrounded value of B:

$$\frac{dT}{dt} = -8(0.0063927...)e^{-0.0063927...(2)}$$
$$\frac{dT}{dt} = -0.051141... \times e^{-0.012785...}$$
$$\frac{dT}{dt} = -0.051141... \times 0.98729...$$
$$\frac{dT}{dt} \approx -0.05049...$$

- The rate of change is approximately $\boxed{-0.0505^\circ\ \mathrm{C}\cdot\mathrm{min}^{-1}}$ (to 3 significant figures).



4. Explanation regarding the temperature limit - As time t increases ($t \to \infty$), the exponential term e^{-Bt} approaches zero because $B > 0$. - Therefore, the temperature T approaches a **horizontal asymptote**:

$$\lim_{t\to\infty}(10 + 8e^{-Bt}) = 10 + 0 = 10$$

- Since $8e^{-Bt}$ is always positive for all finite t, T will always be greater than 10° C. Thus, according to the model, the temperature cannot fall to 5° C because the minimum theoretical temperature (the ambient room temperature) is 10° C.

WMA11_P3(IAL)_Winter_2024_Q6

Solution

The function is given by $f(x) = 2e^{3\sin x}\cos x$ for $0 \leq x \leq 2\pi$.

1. Coordinates of point R The point R is an x-intercept where the curve touches or crosses the x-axis. We set $f(x) = 0$:

$$2e^{3\sin x}\cos x = 0$$

Since the exponential term $e^{3\sin x}$ is always positive for all real x, we must have:

$$\cos x = 0$$

Within the interval $0 \leq x \leq 2\pi$, the solutions for $\cos x = 0$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Looking at Figure 1, the curve crosses the axis at an earlier point (which would be $\frac{\pi}{2}$) and then reaches a minimum at Q. The point R is located further to the right where the curve appears to have a **stationary point** on the axis. Based on the sequence of features in the graph:

- The first root is at $x = \frac{\pi}{2}$.
- The second root is at $x = \frac{3\pi}{2}$. From the sketch, R is the second intercept. Thus, $x = \frac{3\pi}{2}$. The coordinates of R are $(\frac{3\pi}{2}, 0)$.

2. Equation for turning points P and Q To find the **turning points**, we calculate the derivative $f'(x)$ using the **product rule** and the **chain rule**. Let $u = 2e^{3\sin x}$ and $v = \cos x$.

- $\frac{du}{dx} = 2e^{3\sin x} \cdot (3\cos x) = 6e^{3\sin x}\cos x$
- $\frac{dv}{dx} = -\sin x$

Applying the product rule $f'(x) = u\frac{dv}{dx} + v\frac{du}{dx}$:

$$\begin{aligned} f'(x) &= (2e^{3\sin x})(-\sin x) + (\cos x)(6e^{3\sin x}\cos x) \\ &= 2e^{3\sin x}(3\cos^2 x - \sin x) \end{aligned}$$

At turning points, $f'(x) = 0$. Since $2e^{3\sin x} \neq 0$:

$$3\cos^2 x - \sin x = 0$$

Using the **trigonometric identity** $\cos^2 x = 1 - \sin^2 x$:

$$\begin{gathered} 3(1 - \sin^2 x) - \sin x = 0 \\ 3 - 3\sin^2 x - \sin x = 0 \\ 3\sin^2 x + \sin x - 3 = 0 \end{gathered}$$

Comparing this to $a\sin^2 x + b\sin x + c = 0$, we find the integers: $a = 3, b = 1, c = -3$.

3. x-coordinate of point Q We solve the **quadratic equation** $3\sin^2 x + \sin x - 3 = 0$ for $\sin x$ using the quadratic formula:

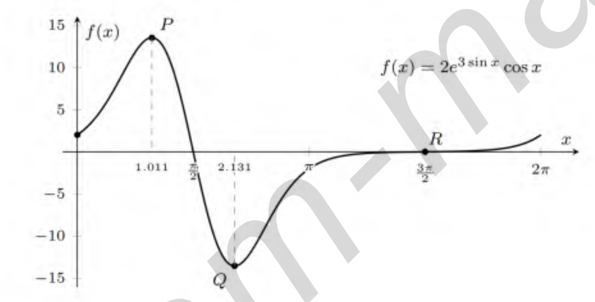
$$\sin x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-3)}}{2(3)}$$
$$= \frac{-1 \pm \sqrt{37}}{6}$$

Calculating the two possible values for $\sin x$:

- $\sin x = \frac{-1+\sqrt{37}}{6} \approx 0.8471$
- $\sin x = \frac{-1-\sqrt{37}}{6} \approx -1.1805$ (Invalid, as $|\sin x| \leq 1$)

Thus, $\sin x = \frac{-1+\sqrt{37}}{6}$. This gives two possible values for x in the range $[0, 2\pi]$:

- $x_1 = \arcsin(0.847127...) \approx 1.011\,\text{rad}$
- $x_2 = \pi - \arcsin(0.847127...) \approx 2.131\,\text{rad}$



From Figure 1, point P is the first turning point (maximum) and point Q is the second turning point (minimum). Therefore, the x-coordinate of Q is the larger value:

$$x_Q = \pi - \arcsin\left(\frac{-1+\sqrt{37}}{6}\right)$$
$$\approx 3.14159 - 1.01078$$
$$\approx 2.13081$$

Rounding to 3 decimal places, we obtain $x = 2.131$.

(a) $(\frac{3\pi}{2}, 0)$
(b) $3\sin^2 x + \sin x - 3 = 0$
(c) $\boxed{2.131}$

WMA11_P3(IAL)_Winter_2024_Q7

Solution

1. Differentiation of the curve equation

The equation of the curve C is given by:

$$y = \frac{16}{9(3x-k)} = \frac{16}{9}(3x-k)^{-1}$$

To find $\frac{dy}{dx}$, we apply the **chain rule**:

$$\begin{aligned}\frac{dy}{dx} &= \frac{16}{9} \cdot (-1)(3x-k)^{-2} \cdot \frac{d}{dx}(3x-k) \\ &= -\frac{16}{9}(3x-k)^{-2} \cdot 3 \\ &= -\frac{16}{3(3x-k)^2}\end{aligned}$$

$$\boxed{\frac{dy}{dx} = -\frac{16}{3(3x-k)^2}}$$

2. Finding the possible values of k

The gradient of the curve at point P (where $x = 1$) is given as -12. Substituting $x = 1$ into the derivative expression:

$$\begin{aligned}-12 &= -\frac{16}{3(3(1)-k)^2} \\ 12 &= \frac{16}{3(3-k)^2} \\ 36(3-k)^2 &= 16 \\ (3-k)^2 &= \frac{16}{36} \\ (3-k)^2 &= \frac{4}{9}\end{aligned}$$

Taking the square root of both sides:

$$\begin{aligned}3-k &= \pm\frac{2}{3} \\ k &= 3 \mp \frac{2}{3}\end{aligned}$$

The two possible values for k are:

- $k = 3 - \frac{2}{3} = \frac{7}{3}$
- $k = 3 + \frac{2}{3} = \frac{11}{3}$

$$\boxed{k = \frac{7}{3}, \frac{11}{3}}$$

3. Equation of the normal to C at P

Given $k < 3$, we use $k = \frac{7}{3}$. First, find the y-coordinate of P at $x = 1$:

$$\begin{aligned} y &= \frac{16}{9(3(1) - \frac{7}{3})} \\ &= \frac{16}{9(3 - \frac{7}{3})} \\ &= \frac{16}{9(\frac{2}{3})} \\ &= \frac{16}{6} = \frac{8}{3} \end{aligned}$$

The gradient of the tangent at P is $m_t = -12$. The gradient of the **normal** m_n is the negative reciprocal:

$$m_n = -\frac{1}{m_t} = \frac{1}{12}$$

Using the point-slope form $y - y_1 = m_n(x - x_1)$:

$$\begin{aligned} y - \frac{8}{3} &= \frac{1}{12}(x - 1) \\ 12y - 32 &= x - 1 \\ x - 12y + 31 &= 0 \end{aligned}$$

$\boxed{x - 12y + 31 = 0}$

4. Algebraic integration

We evaluate the integral using the value $k = \frac{7}{3}$ (as specified by $k < 3$ in the previous part):

$$I = \int_1^3 \frac{16}{9(3x - \frac{7}{3})} dx$$

Simplify the integrand:

$$\begin{aligned} \frac{16}{9(3x - \frac{7}{3})} &= \frac{16}{27x - 21} \\ &= \frac{16}{3(9x - 7)} \end{aligned}$$

Perform the integration using the **natural logarithm** rule:

$$
\begin{aligned}
I &= \frac{16}{3}\int_1^3 \frac{1}{9x-7}dx \\
&= \frac{16}{3}\left[\frac{1}{9}\ln|\ 9x-7\ |\right]_1^3 \\
&= \frac{16}{27}(\ln|\ 9(3)-7\ | - \ln|\ 9(1)-7\ |) \\
&= \frac{16}{27}(\ln 20 - \ln 2) \\
&= \frac{16}{27}\ln\left(\frac{20}{2}\right) \\
&= \frac{16}{27}\ln 10
\end{aligned}
$$

Comparing this to the form $\lambda \ln 10$, we find $\lambda = \frac{16}{27}$.

$$\boxed{\lambda = \frac{16}{27}}$$

WMA11_P3(IAL)_Winter_2024_Q8

Solution

The graph in Figure 2 is defined by the **absolute value function** equation:

$$y = a - |\ 2x - b\ |$$

where a and b are positive constants and $a > b$.

1. Analysis of the graph $y = a - |\ 2x - b\ |$

- **(i) Maximum point** The maximum value of the function occurs when the subtracted term $|\ 2x - b\ |$ is at its minimum. Since the minimum value of an absolute value expression is 0:

$$2x - b = 0$$
$$x = \frac{b}{2}$$

  Substituting $x = \frac{b}{2}$ into the equation gives $y = a - 0 = a$. The coordinates of the maximum point are $\boxed{(\frac{b}{2}, a)}$.

- **(ii) Intersection with the y-axis** The y-intercept occurs where $x = 0$:

$$y = a - |\ 2(0) - b\ |$$
$$= a - |-b\ |$$

  Since $b > 0$, $|-b\ | = b$. Thus, $y = a - b$. The coordinates of the y-intercept are $\boxed{(0, a-b)}$.

- **(iii) Intersections with the x-axis** The x-intercepts occur where $y = 0$:

$$0 = a - |\ 2x - b\ |$$
$$|\ 2x - b\ | = a$$

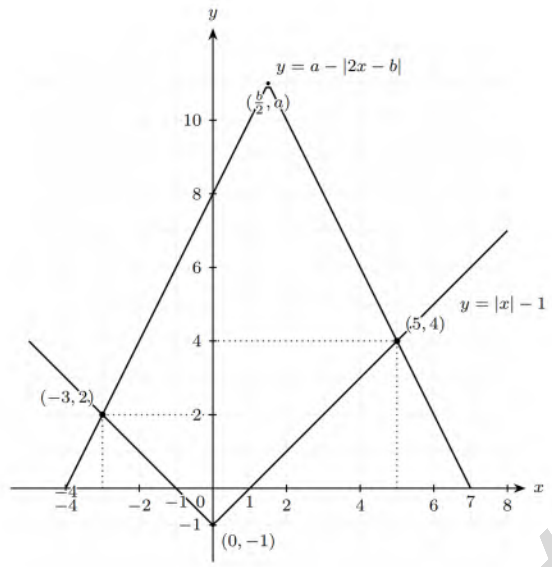
  This yields two linear equations:

$$2x - b = a \quad \Rightarrow \quad 2x = a + b \quad \Rightarrow \quad x = \frac{a+b}{2}$$
$$2x - b = -a \quad \Rightarrow \quad 2x = b - a \quad \Rightarrow \quad x = \frac{b-a}{2}$$

  The coordinates of the x-intercepts are $\boxed{(\frac{b-a}{2}, 0) \text{ and } (\frac{a+b}{2}, 0)}$.

2. Sketch of $y = |\ x\ | - 1$

The graph of $y = |\ x\ | - 1$ is a V-shaped graph with its vertex at $(0, -1)$ and x-intercepts at $(-1, 0)$ and $(1, 0)$.



3. Finding the values of a and b

The graphs $y = |\ x\ | - 1$ and $y = a - |\ 2x - b\ |$ intersect at $x = -3$ and $x = 5$.

- **Step 1: Determine the y-coordinates at the intersection points.** Using $y = |\ x\ | - 1$: At $x = -3$: $y = |-3\ | - 1 = 3 - 1 = 2$. At $x = 5$: $y = |\ 5\ | - 1 = 5 - 1 = 4$. The intersection points are $(-3, 2)$ and $(5, 4)$.
- **Step 2: Substitute these points into $y = a - |\ 2x - b\ |$.** For $(-3, 2)$:

$$2 = a - |\ 2(-3) - b\ | \implies 2 = a - |-6 - b\ |$$

  Since $b > 0$, $-6 - b$ is negative, so $|-6 - b\ | = -(-6 - b) = 6 + b$.

$$2 = a - (6 + b) \implies a - b = 8 \quad \text{— (Eq. 1)}$$

  For $(5, 4)$:

$$4 = a - |\ 2(5) - b\ | \implies 4 = a - |\ 10 - b\ |$$

  From the graph in Figure 2, the peak is at $x = \frac{b}{2}$. Since the intersection $x = 5$ is to the right of the peak (based on the slope change), we assume $10 - b > 0$.

$$4 = a - (10 - b) \implies a + b = 14 \quad \text{— (Eq. 2)}$$

- **Step 3: Solve the system of equations.** Adding (Eq. 1) and (Eq. 2):

$$(a - b) + (a + b) = 8 + 14$$
$$2a = 22$$
$$a = 11$$

  Substituting $a = 11$ into (Eq. 2):

$$11 + b = 14$$
$$b = 3$$

  (Check: $a > b$ holds as $11 > 3$, and $10 - b = 7 > 0$ matches our assumption).

$\boxed{a = 11, b = 3}$

WMA11_P3(IAL)_Winter_2024_Q9

Solution

1. Derivation of the simplified form

To show that the given equation can be written in the form $3\sin 2\theta - 4\cos 2\theta = 2$, we begin by manipulating the original expression:

$$\frac{3\sin\theta\cos\theta}{\cos\theta + \sin\theta} = (2 + \sec 2\theta)(\cos\theta - \sin\theta)$$

- Multiply both sides by $(\cos\theta + \sin\theta)$ to clear the denominator:

$$3\sin\theta\cos\theta = (2 + \sec 2\theta)(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$$

- Apply the **difference of squares** identity, $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta) = \cos^2\theta - \sin^2\theta$.
- Use the **double angle identity** for cosine: $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
- Use the double angle identity for sine: $\sin 2\theta = 2\sin\theta\cos\theta$, which implies $3\sin\theta\cos\theta = \frac{3}{2}\sin 2\theta$.

Substituting these into the equation:

$$\frac{3}{2}\sin 2\theta = (2 + \sec 2\theta)\cos 2\theta$$

- Expand the right-hand side using the definition $\sec 2\theta = \frac{1}{\cos 2\theta}$:

$$\frac{3}{2}\sin 2\theta = 2\cos 2\theta + \sec 2\theta\cos 2\theta$$
$$\frac{3}{2}\sin 2\theta = 2\cos 2\theta + 1$$

- Multiply the entire equation by 2 to eliminate the fraction:

$$3\sin 2\theta = 4\cos 2\theta + 2$$
$$3\sin 2\theta - 4\cos 2\theta = 2$$

2. Solving the equation for $\pi < x < \frac{3\pi}{2}$

We solve $3\sin 2x - 4\cos 2x = 2$ using the **R-formula** method, where $a\sin A - b\cos A = R\sin(A - \alpha)$.

- Calculate R:

$$R = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

- Calculate the phase angle α:

$$\tan\alpha = \frac{4}{3} \implies \alpha = \arctan\left(\frac{4}{3}\right) \approx 0.927295\,\text{rad}$$

- Rewrite the equation:

$$5\sin(2x - 0.927295) = 2$$

$$\sin(2x - 0.927295) = \frac{2}{5} = 0.4$$

- Determine the range for the transformed variable $u = 2x - \alpha$: Given $\pi < x < \frac{3\pi}{2}$, then $2\pi < 2x < 3\pi$. Subtracting α: $2\pi - 0.927295 < 2x - 0.927295 < 3\pi - 0.927295$.

$$5.35589 < 2x - 0.927295 < 8.49748$$

- Find the solutions for $\sin u = 0.4$ within this range: The principal value is $u_p = \arcsin(0.4) \approx 0.411517\,\text{rad}$. The general solutions for $\sin u = 0.4$ are $u = 2n\pi + u_p$ or $u = (2n+1)\pi - u_p$.
  - For $n = 1$: $u = 2\pi + 0.411517 \approx 6.69470\,\text{rad}$ (Within range)
  - For $n = 1$: $u = 3\pi - 0.411517 \approx 9.01326\,\text{rad}$ (Outside range)
- Solve for x:

$$\begin{aligned} 2x - 0.927295 &= 6.69470 \\ 2x &= 7.621995 \\ x &= 3.8109975 \end{aligned}$$

Rounding to 3 significant figures:

$$\boxed{x = 3.81}$$

WMA11_P3(IAL)_Summer_2025_Q1

Solution

1. Evaluation of the composite function $gf(1)$

- First, evaluate the inner function $f(x)$ at $x = 1$:

$$f(1) = \frac{2(1)}{3(1)+1} = \frac{2}{4} = 0.5$$

- Next, substitute this result into the outer function $g(x)$:

$$gf(1) = g(0.5) = 4 - (0.5)^2 = 4 - 0.25 = 3.75$$

$\boxed{3.75}$

2. Determination of the range of f

- The function is defined as $f(x) = \frac{2x}{3x+1}$ for $x \geq 0$.
- At the lower bound of the domain, $x = 0$:

$$f(0) = \frac{0}{1} = 0$$

- To find the behavior as $x \to \infty$, we evaluate the **horizontal asymptote**:

$$\lim_{x\to\infty} \frac{2x}{3x+1} = \lim_{x\to\infty} \frac{2}{3+\frac{1}{x}} = \frac{2}{3}$$

- Since $f'(x) = \frac{2(3x+1)-3(2x)}{(3x+1)^2} = \frac{2}{(3x+1)^2} > 0$, the function is strictly increasing. Thus, the range starts at 0 and approaches $2/3$ but never reaches it. $\boxed{0 \leq f(x) < \frac{2}{3}}$

3. Finding the inverse function $f^{-1}(x)$

- Let $y = f(x)$ and solve for x in terms of y:

$$\begin{aligned} y &= \frac{2x}{3x+1} \\ y(3x+1) &= 2x \\ 3xy + y &= 2x \\ y &= 2x - 3xy \\ y &= x(2-3y) \\ x &= \frac{y}{2-3y} \end{aligned}$$

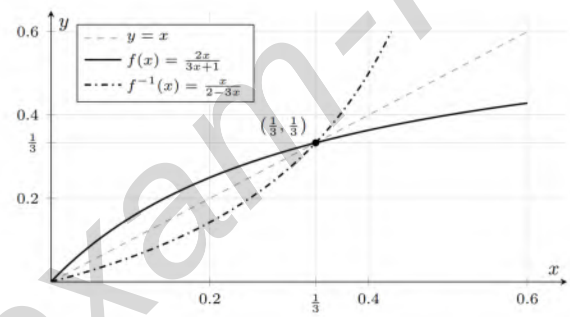
- Swapping the variables to express the **inverse function** in terms of x: $\boxed{f^{-1}(x) = \dfrac{x}{2-3x}}$

4. Solving the equation $f^{-1}(x) = f(x)$

- For a function $f(x)$ that is strictly increasing, the intersection of $f(x)$ and $f^{-1}(x)$ occurs on the line $y = x$. We set $f(x) = x$:

$$\frac{2x}{3x+1} = x$$
$$2x = x(3x+1)$$
$$2x = 3x^2 + x$$
$$3x^2 - x = 0$$
$$x(3x-1) = 0$$

- Solving for x gives $x = 0$ or $x = 1/3$.
- We verify if these values lie within the domain ($x \geq 0$) and the range ($0 \leq y < 2/3$) of the original function. Both 0 and $1/3$ are valid.



$\boxed{x = 0, \dfrac{1}{3}}$

WMA11_P3(IAL)_Summer_2025_Q2

Solution

1. Expressing $f(x)$ in the form $R\cos(x+\alpha)$

To express $f(x) = 7\cos x - 24\sin x$ in the form $R\cos(x+\alpha)$, we use the **harmonic addition theorem**. Expanding the target form using the **cosine addition formula**:

$$R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$$

By comparing the coefficients with $f(x) = 7\cos x - 24\sin x$, we obtain the following system of equations:

$$R\cos\alpha = 7$$
$$R\sin\alpha = 24$$

- **Finding R:** Squaring and adding the equations:

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 7^2 + 24^2$$
$$R^2(\cos^2\alpha + \sin^2\alpha) = 49 + 576$$
$$R^2 = 625$$
$$R = \sqrt{625} = 25$$

- **Finding α:** Dividing the sine equation by the cosine equation:

$$\tan\alpha = \frac{24}{7}$$
$$\alpha = \arctan\left(\frac{24}{7}\right)$$

  Using a calculator to find the value in radians (to 3 decimal places):

$$\alpha \approx 1.287\,\mathrm{rad}$$

Thus, $f(x) = 25\cos(x + 1.287)$.

2. Finding the minimum value of $g(x)$

The function is given by:

$$g(x) = \frac{5}{90 - 3f(2x)}$$

From part (a), we know $f(2x) = 25\cos(2x+\alpha)$. Substituting this into $g(x)$:

$$g(x) = \frac{5}{90 - 3[25\cos(2x+\alpha)]} = \frac{5}{90 - 75\cos(2x+\alpha)}$$

- **(i) Minimum value of $g(x)$:** To minimize the fraction $g(x)$, we must maximize the denominator $D = 90 - 75\cos(2x+\alpha)$. The denominator is maximized when $\cos(2x+\alpha)$ takes its minimum value, which is -1.

$$g_{\min} = \frac{5}{90 - 75(-1)} = \frac{5}{90 + 75} = \frac{5}{165} = \frac{1}{33}$$

- **(ii) Smallest positive value of x for the minimum:** The minimum occurs when $\cos(2x + \alpha) = -1$. The general solution for this is:

$$2x + \alpha = \pi, 3\pi, 5\pi, ...$$

Using the exact value $\alpha = \arctan(24/7)$:

$$2x + 1.287002... = \pi$$
$$2x = 3.141592... - 1.287002...$$
$$2x = 1.854590...$$
$$x = 0.927295...$$

Rounding to 3 decimal places, we get $x \approx 0.927\,\text{rad}$.

Final Answers:

(a) $R = 25$, $\alpha = 1.287\,\text{rad}$

(b) (i) $\boxed{\frac{1}{33}}$

(c) (ii) $\boxed{0.927}$

WMA11_P3(IAL)_Summer_2025_Q3

Solution

1. Finding the equation linking $\log_{10} y$ and $\log_{10} x$

The graph shows a **linear relationship** between the variables $Y = \log_{10} y$ and $X = \log_{10} x$. The line passes through the points $(-3.5, 0)$ and $(0, -2)$.

- **Calculate the gradient (m):** Using the **gradient formula** $m = \frac{Y_2 - Y_1}{X_2 - X_1}$:

$$\begin{aligned} m &= \frac{-2-0}{0-(-3.5)} \\ &= \frac{-2}{3.5} \\ &= -\frac{\frac{2}{7}}{2} \\ &= -\frac{4}{7} \end{aligned}$$

- **Identify the vertical intercept (c):** The line crosses the vertical axis at $(0, -2)$, so the intercept is $c = -2$.
- **Formulate the linear equation:** Using the **slope-intercept form** $Y = mX + c$:

$$\log_{10} y = -\frac{4}{7} \log_{10} x - 2$$

2. Expressing y in the form px^q

To find the relationship between y and x, we apply **logarithmic identities** to the equation derived in part (a).

- **Apply the power law of logarithms:**

$$\log_{10} y = \log_{10}(x^{-4/7}) - 2$$

- **Express the constant as a logarithm:** Since $2 = \log_{10}(10^2) = \log_{10} 100$:

$$\log_{10} y = \log_{10}(x^{-4/7}) - \log_{10} 100$$

- **Apply the quotient law of logarithms:**

$$\log_{10} y = \log_{10}\left(\frac{x^{-4/7}}{100}\right)$$

- **Remove logarithms by taking the antilogarithm of both sides:**

$$\begin{aligned} y &= \frac{1}{100}x^{-4/7} \\ &= 0.01x^{-4/7} \end{aligned}$$

Comparing this to the form $y = px^q$, we identify the **rational constants** as $p = \frac{1}{100}$ and $q = -\frac{4}{7}$.

Final Answer: (a) $\log_{10} y = -\frac{4}{7}\log_{10} x - 2$ (b) $y = \frac{1}{100}x^{-4/7}$

WMA11_P3(IAL)_Summer_2025_Q4

Solution

1. Simplification of the function $f(x)$

To simplify the given function $f(x) = \frac{49x}{x^2+x-12} + \frac{7x}{x+4}$, we first factor the quadratic expression in the denominator of the first term.

- **Factor the denominator:** The quadratic $x^2 + x - 12$ can be factored by finding two numbers that multiply to -12 and add to 1. These numbers are 4 and -3.

$$x^2 + x - 12 = (x+4)(x-3)$$

- **Combine the fractions:** We rewrite the function using a **common denominator**, which is $(x+4)(x-3)$.

$$\begin{aligned} f(x) &= \frac{49x}{(x+4)(x-3)} + \frac{7x}{x+4} \\ &= \frac{49x}{(x+4)(x-3)} + \frac{7x(x-3)}{(x+4)(x-3)} \\ &= \frac{49x + 7x(x-3)}{(x+4)(x-3)} \end{aligned}$$

- **Simplify the numerator:**

$$\begin{aligned} 49x + 7x^2 - 21x &= 7x^2 + 28x \\ &= 7x(x+4) \end{aligned}$$

- **Final simplification:** Substitute the simplified numerator back into the expression:

$$f(x) = \frac{7x(x+4)}{(x+4)(x-3)}$$

Since $x > 3$, the term $(x+4)$ is non-zero, allowing us to cancel it:

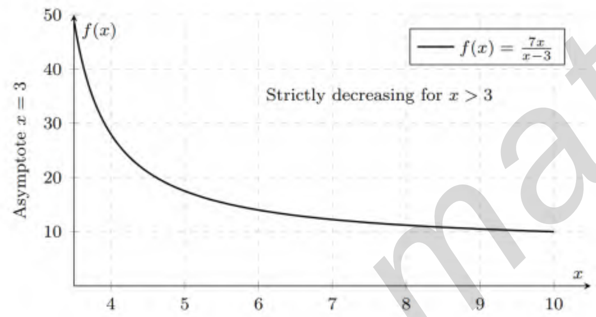
$$\boxed{f(x) = \frac{7x}{x-3}}$$

2. Differentiation of $f(x)$

To find $f'(x)$, we apply the **quotient rule** for differentiation, which states that for a function u/v, the derivative is $\frac{u'v-uv'}{v^2}$.

- **Identify components:** Let $u = 7x$ and $v = x - 3$. Then $u' = 7$ and $v' = 1$.
- **Apply the rule:**

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{7x}{x-3}\right) \\ &= \frac{(7)(x-3)-(7x)(1)}{(x-3)^2} \\ &= \frac{7x-21-7x}{(x-3)^2} \\ &= \frac{-21}{(x-3)^2} \end{aligned}$$



The derivative in its simplest form is:

$$\boxed{f'(x) = -\frac{21}{(x-3)^2}}$$

WMA11_P3(IAL)_Summer_2025_Q5

Solution

To evaluate the given indefinite integrals, we apply standard integration techniques such as trigonometric identities and the method of substitution.

1. Evaluation of $\int \sin^2 3x \, dx$

- To integrate $\sin^2 3x$, we use the **power-reduction identity** derived from the **double-angle formula** for cosine:

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

- Substituting $\theta = 3x$ into the identity:

$$\sin^2 3x = \frac{1 - \cos 6x}{2}$$

- Now, substitute this expression back into the integral:

$$\begin{aligned} \int \sin^2 3x \, dx &= \int \frac{1 - \cos 6x}{2} \, dx \\ &= \frac{1}{2} \int (1 - \cos 6x) \, dx \\ &= \frac{1}{2} \left(\int 1 \, dx - \int \cos 6x \, dx \right) \\ &= \frac{1}{2} \left(x - \frac{1}{6} \sin 6x \right) + C \\ &= \frac{1}{2} x - \frac{1}{12} \sin 6x + C \end{aligned}$$

where C is the **constant of integration**.

2. Evaluation of $\int x(x^2 + 4)^{3/2} \, dx$

- We use the method of **u-substitution**. Let:

$$u = x^2 + 4$$

- Differentiating both sides with respect to x:

$$\begin{aligned} \frac{du}{dx} &= 2x \\ du &= 2x \, dx \\ \frac{1}{2} du &= x \, dx \end{aligned}$$

- Substitute u and $\frac{1}{2} du$ into the integral:

$$\int x(x^2+4)^{3/2}\,dx = \int (x^2+4)^{3/2}\cdot(x\,dx)$$
$$= \int u^{3/2}\cdot\frac{1}{2}du$$
$$= \frac{1}{2}\int u^{3/2}\,du$$

- Apply the **power rule for integration**, $\int u^n\,du = \frac{u^{n+1}}{n+1}+C$:

$$\frac{1}{2}\int u^{3/2}\,du = \frac{1}{2}\left(\frac{u^{3/2+1}}{3/2+1}\right)+C$$
$$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2}\right)+C$$
$$= \frac{1}{2}\cdot\frac{2}{5}u^{5/2}+C$$
$$= \frac{1}{5}u^{5/2}+C$$

- Back-substitute $u = x^2+4$:

$$\frac{1}{5}(x^2+4)^{5/2}+C$$

Final Answers:

(i) $\boxed{\frac{1}{2}x-\frac{1}{12}\sin 6x+C}$

(ii) $\boxed{\frac{1}{5}(x^2+4)^{5/2}+C}$

WMA11_P3(IAL)_Summer_2025_Q6

Solution

The temperature θ of a computer processor t minutes after being switched off is modeled by **Newton's Law of Cooling** in the form:

$$\theta = 21 + Ae^{-kt}$$

1. Finding the value of A The computer is switched off at $t = 0$. We are given that the initial temperature is 75° C. - Substitute $t = 0$ and $\theta = 75$ into the model:

$$75 = 21 + Ae^{-k(0)}$$
$$75 = 21 + A(1)$$
$$A = 75 - 21$$
$$A = 54$$

$\boxed{A = 54}$

2. Finding the value of k It takes 5 minutes for the temperature to decrease to 25° C. Thus, at $t = 5$, $\theta = 25$. - Substitute $A = 54$, $t = 5$, and $\theta = 25$ into the equation:

$$25 = 21 + 54e^{-5k}$$
$$4 = 54e^{-5k}$$
$$e^{-5k} = \frac{4}{54} = \frac{2}{27}$$

- Take the **natural logarithm** of both sides:

$$-5k = \ln\left(\frac{2}{27}\right)$$
$$k = -\frac{1}{5}\ln\left(\frac{2}{27}\right)$$
$$k \approx 0.520576...$$

- Rounding to 3 significant figures: $\boxed{k = 0.521}$

3. Finding the value of T The rate of change of temperature is given by the **derivative** $d\theta/dt$. We are told that at $t = T$, the temperature is decreasing at a rate of 9° C per minute, which means $d\theta/dt = -9$. - Differentiate the model with respect to t:

$$\frac{d\theta}{dt} = \frac{d}{dt}(21 + 54e^{-kt})$$
$$= -54ke^{-kt}$$

- Set the rate to -9 at $t = T$:

$$-9 = -54ke^{-kT}$$
$$e^{-kT} = \frac{9}{54k} = \frac{1}{6k}$$

- Using the exact value of $k = \frac{1}{5}\ln(27/2)$:

$$-kT = \ln\left(\frac{1}{6k}\right)$$

$$T = -\frac{1}{k}\ln\left(\frac{1}{6k}\right) = \frac{\ln(6k)}{k}$$

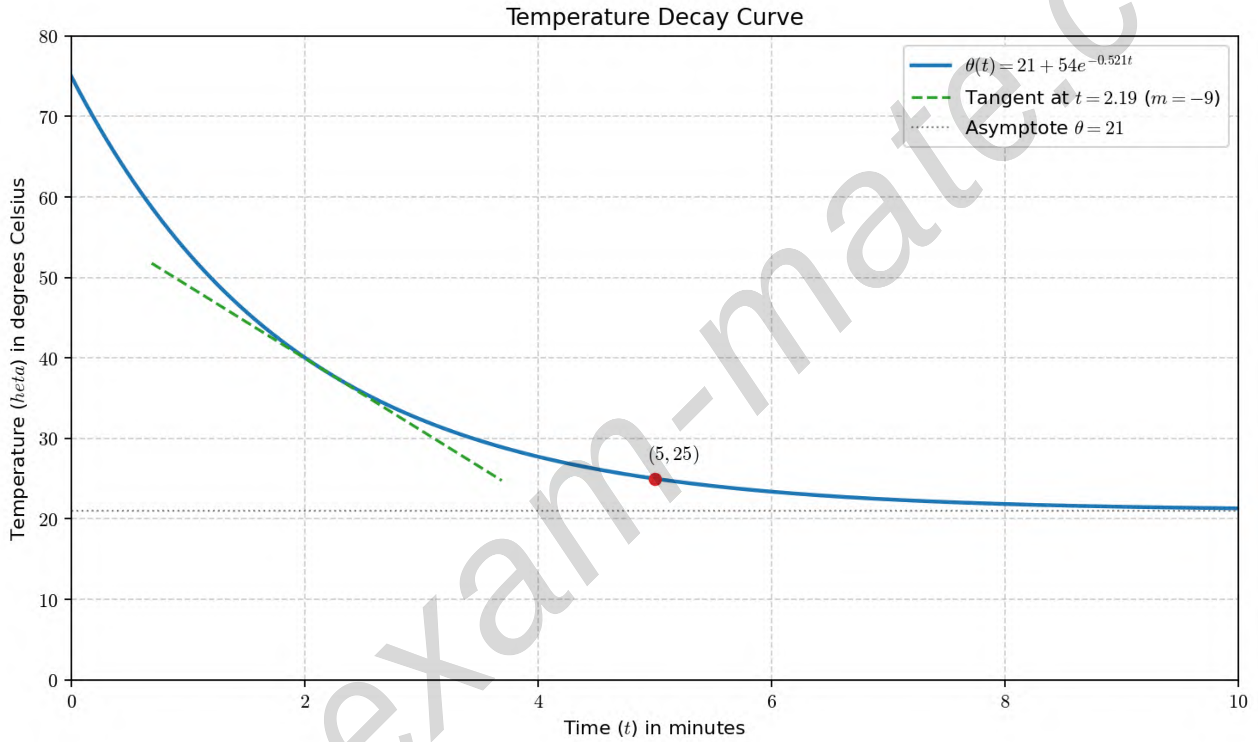
- Substitute $k \approx 0.520576$:

$$T = \frac{\ln(6 \cdot 0.520576...)}{0.520576...}$$

$$T \approx \frac{\ln(3.12345...)}{0.520576...}$$

$$T \approx 2.1878...$$

- Rounding to 2 decimal places: $\boxed{T = 2.19}$



WMA11_P3(IAL)_Summer_2025_Q7

Solution

The problem involves analyzing a continuous curve to find its **stationary point** and approximating its x-coordinate using numerical methods.

1. Derivation of the Iterative Equation

The curve is defined by the equation:

$$y = e^{-x^2}\sin(3x), \quad 0 \le x \le \frac{\pi}{3}$$

To find the x-coordinate of the stationary point P, we apply the **product rule** and the **chain rule** to find the first derivative dy/dx:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^{-x^2})\cdot\sin(3x) + e^{-x^2}\cdot\frac{d}{dx}(\sin(3x)) \\ &= -2xe^{-x^2}\sin(3x) + 3e^{-x^2}\cos(3x)\end{aligned}$$

At a stationary point, the gradient is zero ($dy/dx = 0$):

$$\begin{aligned}-2xe^{-x^2}\sin(3x) + 3e^{-x^2}\cos(3x) &= 0 \\ e^{-x^2}(3\cos(3x) - 2x\sin(3x)) &= 0\end{aligned}$$

Since $e^{-x^2} \neq 0$ for all real x, we must have:

$$\begin{aligned}3\cos(3x) - 2x\sin(3x) &= 0 \\ 3\cos(3x) &= 2x\sin(3x) \\ \frac{\sin(3x)}{\cos(3x)} &= \frac{3}{2x} \\ \tan(3x) &= \frac{3}{2x}\end{aligned}$$

Taking the **arctangent** of both sides:

$$\begin{aligned}3x &= \arctan\left(\frac{3}{2x}\right) \\ x &= \frac{1}{3}\arctan\left(\frac{3}{2x}\right)\end{aligned}$$

2. Iterative Calculation

Using the **fixed-point iteration** formula $x_{n+1} = \frac{1}{3}\arctan\left(\frac{3}{2x_n}\right)$ with an initial value $x_1 = 0.4$:

- **(i) Find x_2:**

  $$x_2 = \frac{1}{3}\arctan\left(\frac{3}{2(0.4)}\right) = \frac{1}{3}\arctan(3.75) \approx 0.43685...$$

  Rounding to 4 decimal places: $\boxed{x_2 = 0.4369}$

- **(ii) Find x_4:** First, calculate x_3:

$$x_3 = \frac{1}{3}\arctan\left(\frac{3}{2(0.43685...)}\right) \approx 0.42896...$$

Then, calculate x_4:

$$x_4 = \frac{1}{3}\arctan\left(\frac{3}{2(0.42896...)}\right) \approx 0.43064...$$

Rounding to 4 decimal places: $\boxed{x_4 = 0.4306}$

3. Verification of the Root

To show that the x-coordinate of P is 0.430 correct to 3 decimal places, we use the **change of sign** method on the interval $[0.4295, 0.4305]$. We define the function:

$$f(x) = 3\cos(3x) - 2x\sin(3x)$$

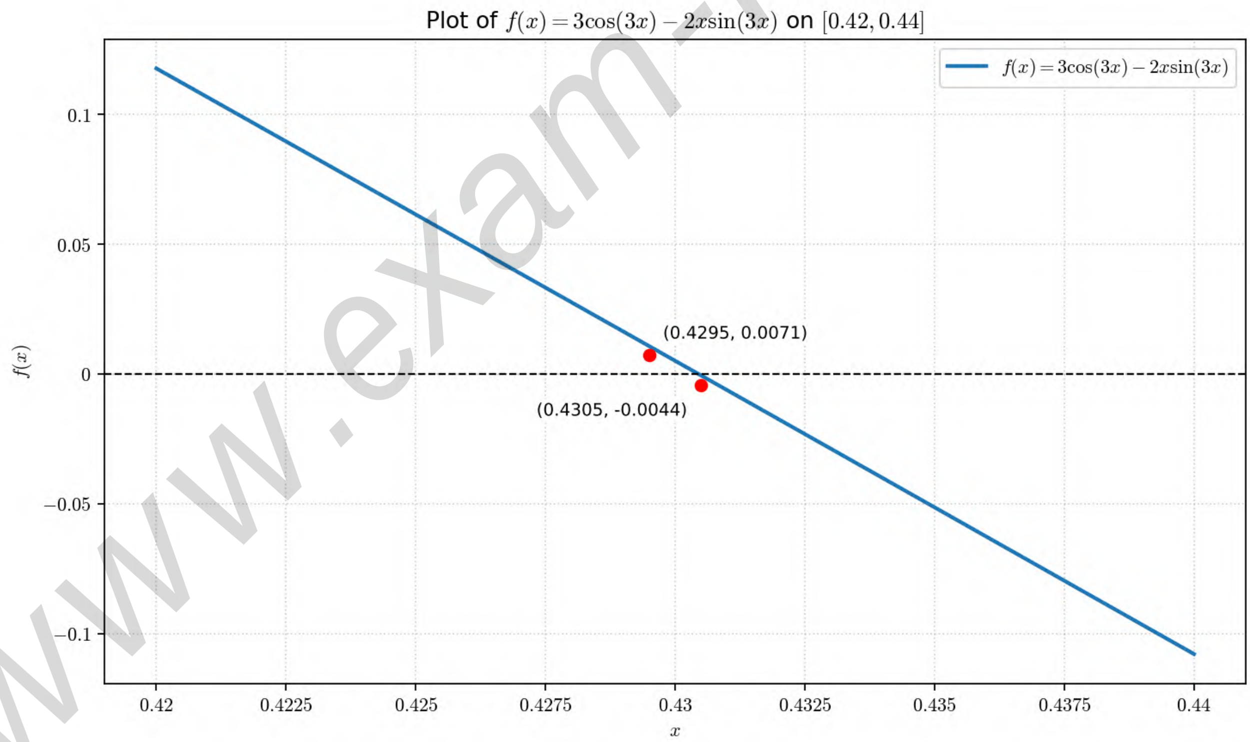
Evaluating $f(x)$ at the boundaries:

- For $x = 0.4295$:

$$f(0.4295) = 3\cos(3 \cdot 0.4295) - 2(0.4295)\sin(3 \cdot 0.4295) \approx 0.0071 > 0$$

- For $x = 0.4305$:

$$f(0.4305) = 3\cos(3 \cdot 0.4305) - 2(0.4305)\sin(3 \cdot 0.4305) \approx -0.0044 < 0$$



Since $f(x)$ is continuous and there is a change of sign between $x = 0.4295$ and $x = 0.4305$, by the **Intermediate Value Theorem**, there exists a root α such that $0.4295 < \alpha < 0.4305$. Any value in this interval rounds to 0.430 to 3 decimal places.

$\boxed{x \approx 0.430 \quad \text{(to 3 d.p.)}}$

WMA11_P3(IAL)_Summer_2025_Q8

Solution

1. Proof of the Triple Angle Identity for Tangent

To prove the identity for $\tan 3x$, we utilize the **angle addition formula** for tangent:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

- **Step 1: Express $\tan 3x$ in terms of $2x$ and x**

$$\tan 3x = \tan(2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

- **Step 2: Substitute the double angle formula** Recall the **double angle formula** $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$. Substituting this into the expression:

$$\begin{aligned}\tan 3x &= \frac{\frac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \left(\frac{2\tan x}{1-\tan^2 x}\right)\tan x} \\ &= \frac{\frac{2\tan x + \tan x(1-\tan^2 x)}{1-\tan^2 x}}{\frac{1-\tan^2 x - 2\tan^2 x}{1-\tan^2 x}} \\ &= \frac{2\tan x + \tan x - \tan^3 x}{1 - 3\tan^2 x} \\ &= \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\end{aligned}$$

  This completes the proof for $x \neq (2n+1)\frac{\pi}{6}$, where the denominator is non-zero.

2. Solving the Equation

Given the equation for $0 < \theta < \frac{\pi}{2}$:

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} = 2\sec^2 3\theta - 8$$

- **Step 1: Simplify using the identity from part (a)** The left-hand side is exactly $\tan 3\theta$. Thus:

$$\tan 3\theta = 2\sec^2 3\theta - 8$$

- **Step 2: Convert to a single trigonometric function** Using the **Pythagorean identity** $\sec^2 A = 1 + \tan^2 A$:

$$\begin{aligned}\tan 3\theta &= 2(1 + \tan^2 3\theta) - 8 \\ \tan 3\theta &= 2 + 2\tan^2 3\theta - 8 \\ 0 &= 2\tan^2 3\theta - \tan 3\theta - 6\end{aligned}$$

- **Step 3: Solve the quadratic equation** Let $u = \tan 3\theta$. The equation becomes $2u^2 - u - 6 = 0$. Factoring the quadratic:

$$(2u+3)(u-2) = 0$$

This gives two possible values for $\tan 3\theta$:

1. $\tan 3\theta = 2$
2. $\tan 3\theta = -1.5$

- **Step 4: Determine the range for 3θ** Since $0 < \theta < \frac{\pi}{2}$, the range for the argument is $0 < 3\theta < \frac{3\pi}{2}$ (approximately $0 < 3\theta < 4.712\,\text{rad}$).
- **Step 5: Calculate values for θ Case 1: $\tan 3\theta = 2$**
    - $3\theta_1 = \arctan(2) \approx 1.1071\,\text{rad}$
    - $3\theta_2 = \arctan(2) + \pi \approx 4.2487\,\text{rad}$
    - $\theta_1 = \frac{1.1071}{3} \approx 0.3690\,\text{rad}$
    - $\theta_2 = \frac{4.2487}{3} \approx 1.4162\,\text{rad}$

  Case 2: $\tan 3\theta = -1.5$
    - $3\theta_3 = \arctan(-1.5) + \pi \approx -0.9828 + 3.1416 = 2.1588\,\text{rad}$
    - $3\theta_4 = \arctan(-1.5) + 2\pi \approx 5.3004\,\text{rad}$ (Outside range $3\theta < 4.712$)
    - $\theta_3 = \frac{2.1588}{3} \approx 0.7196\,\text{rad}$
- **Step 6: Final rounding** Rounding the valid solutions to 2 decimal places:
    - $\theta \approx 0.37$
    - $\theta \approx 0.72$
    - $\theta \approx 1.42$

$\boxed{\theta = 0.37, 0.72, 1.42}$

WMA11_P3(IAL)_Summer_2025_Q9

Solution

The curve C is defined by the equation:

$$x = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right), \quad \frac{\pi}{12} < y < \frac{3\pi}{4}$$

1. Finding the y-coordinates of points A and B Points A and B are the intersections of the curve with the y-axis, where $x = 0$.

$$0 = \frac{2}{3}\sin\left(3y + \frac{\pi}{4}\right)$$
$$\sin\left(3y + \frac{\pi}{4}\right) = 0$$

The general solution for $\sin(\theta) = 0$ is $\theta = n\pi$ for $n \in \mathbb{Z}$.

- For $n = 1$: $3y + \frac{\pi}{4} = \pi \implies 3y = \frac{3\pi}{4} \implies y = \frac{\pi}{4}$
- For $n = 2$: $3y + \frac{\pi}{4} = 2\pi \implies 3y = \frac{7\pi}{4} \implies y = \frac{7\pi}{12}$

Checking the range $\frac{\pi}{12} < y < \frac{3\pi}{4}$:

- $\frac{\pi}{4} = \frac{3\pi}{12}$ (within range)
- $\frac{7\pi}{12}$ (within range, since $\frac{3\pi}{4} = \frac{9\pi}{12}$)

From Figure 2, $y_B > y_A$. Thus:

- Point A: $y = \frac{\pi}{4}$
- Point B: $y = \frac{7\pi}{12}$

2. Showing the derivative relationship Differentiating x with respect to y using the **Chain Rule**:

$$\frac{dx}{dy} = \frac{2}{3} \cdot 3\cos\left(3y + \frac{\pi}{4}\right)$$
$$= 2\cos\left(3y + \frac{\pi}{4}\right)$$

Using the **Inverse Function Theorem**, $\frac{dy}{dx} = \frac{1}{dx/dy}$:

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4\cos^2(3y + \frac{\pi}{4})}$$

Using the identity $\cos^2\theta = 1 - \sin^2\theta$:

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4[1 - \sin^2(3y + \frac{\pi}{4})]}$$

From the original equation, $\sin(3y + \frac{\pi}{4}) = \frac{3x}{2}$. Substituting this:

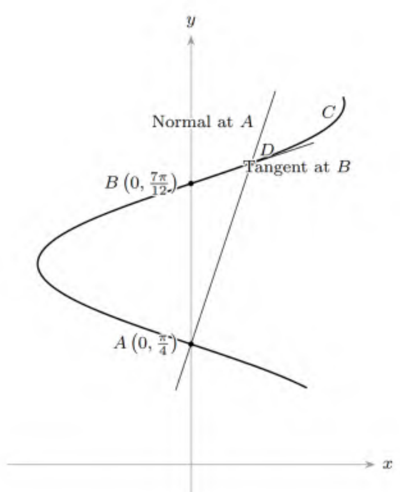
$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4\left[1-\left(\frac{3x}{2}\right)^2\right]}$$
$$= \frac{1}{4\left(1-\frac{9x^2}{4}\right)}$$
$$= \frac{1}{4-9x^2}$$

Comparing this to $\frac{1}{p-qx^2}$, we find $p = 4$ and $q = 9$.

3. Finding the x-coordinate of point D Point D is the intersection of the **normal** at A and the **tangent** at B.

- **At point A ($x = 0, y = \frac{\pi}{4}$):** From part (b), $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4-9(0)^2} = \frac{1}{4}$. At A, $\frac{dx}{dy} = 2\cos(\pi) = -2$, so $\frac{dy}{dx} = -\frac{1}{2}$. The gradient of the normal $m_n = -\frac{1}{dy/dx} = 2$. Equation of normal at A: $y - \frac{\pi}{4} = 2(x-0) \implies y = 2x + \frac{\pi}{4}$.
- **At point B ($x = 0, y = \frac{7\pi}{12}$):** At B, $\frac{dx}{dy} = 2\cos(2\pi) = 2$, so $\frac{dy}{dx} = \frac{1}{2}$. Equation of tangent at B: $y - \frac{7\pi}{12} = \frac{1}{2}(x-0) \implies y = \frac{1}{2}x + \frac{7\pi}{12}$.
- **Intersection point D:** Equating the two expressions for y:

$$2x + \frac{\pi}{4} = \frac{1}{2}x + \frac{7\pi}{12}$$
$$\frac{3}{2}x = \frac{7\pi}{12} - \frac{3\pi}{12}$$
$$\frac{3}{2}x = \frac{4\pi}{12}$$
$$\frac{3}{2}x = \frac{\pi}{3}$$
$$x = \frac{2\pi}{9}$$



The exact x-coordinate of D is:

$$\boxed{\frac{2\pi}{9}}$$

WMA11_P3(IAL)_Summer_2025_Q10

Solution

Based on the provided images, we are given the function $f(x) = \mid kx - 10 \mid + k$, where k is a positive constant. The graph of $y = f(x)$ is a V-shaped **absolute value function**.

1. Finding the coordinates of A and P

- **(i) The y-coordinate of A** The point A is the **y-intercept** of the graph, which occurs when $x = 0$.

$$\begin{aligned} f(0) &= \mid k(0) - 10 \mid + k \\ &= \mid -10 \mid + k \\ &= 10 + k \end{aligned}$$

  The y-coordinate of A is $10 + k$.

- **(ii) The coordinates of P** The point P is the **vertex** of the absolute value graph. The vertex of a function in the form $y = a \mid bx - c \mid + d$ occurs when the expression inside the absolute value is zero.

$$\begin{aligned} kx - 10 &= 0 \\ kx &= 10 \\ x &= \frac{10}{k} \end{aligned}$$

  Substituting $x = \frac{10}{k}$ into $f(x)$:

$$\begin{aligned} f\left(\frac{10}{k}\right) &= \mid k\left(\frac{10}{k}\right) - 10 \mid + k \\ &= \mid 10 - 10 \mid + k \\ &= k \end{aligned}$$

  The coordinates of P are $\left(\frac{10}{k}, k\right)$.

2. Solving the inequality $\mid kx - 10 \mid + k \geq 2k$

Subtract k from both sides:

$$\mid kx - 10 \mid \geq k$$

Since $k > 0$, we can split this **absolute value inequality** into two cases:

- **Case 1:** $kx - 10 \geq k$

$$\begin{aligned} kx &\geq 10 + k \\ x &\geq \frac{10 + k}{k} \\ x &\geq \frac{10}{k} + 1 \end{aligned}$$

- **Case 2:** $kx - 10 \leq -k$

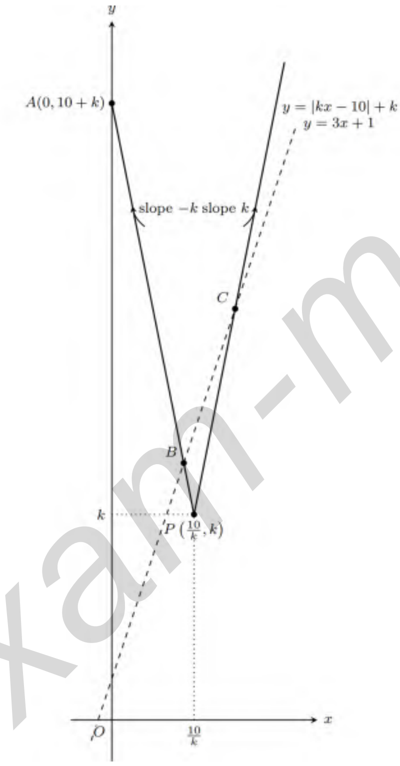
$$kx \leq 10 - k$$
$$x \leq \frac{10 - k}{k}$$
$$x \leq \frac{10}{k} - 1$$

The range of values for x is $x \leq \frac{10}{k} - 1$ or $x \geq \frac{10}{k} + 1$.

3. Finding the range of values for k

The line $y = 3x + 1$ intersects $f(x) = |\ kx - 10\ | + k$ at two distinct points. The function $f(x)$ consists of two linear branches:

- Left branch ($x < \frac{10}{k}$): $y = -(kx - 10) + k = -kx + 10 + k$
- Right branch ($x \geq \frac{10}{k}$): $y = (kx - 10) + k = kx - 10 + k$



For the line $y = 3x + 1$ to intersect the graph at two distinct points, the following conditions must be met:

- **Condition 1: Slope comparison.** The slope of the line ($m = 3$) must be less than the slope of the right branch of the V-shape to ensure they eventually intersect if the line starts “below” the branch, or more specifically, to ensure the line is not parallel to or steeper than the right branch in a way that prevents two intersections. However, looking at the geometry, for two intersections to occur, the line must be “flatter” than the right branch ($3 < k$) and the line must pass “above” the vertex P.
- **Condition 2: Position relative to vertex.** At the x-coordinate of the vertex ($x = \frac{10}{k}$), the y-value of the line must be greater than the y-value of the vertex ($y = k$).

$$3\left(\frac{10}{k}\right) + 1 > k$$
$$\frac{30}{k} + 1 > k$$

Multiply by k (since $k > 0$):

$$30 + k > k^2$$
$$k^2 - k - 30 < 0$$
$$(k-6)(k+5) < 0$$

The roots are $k = 6$ and $k = -5$. Since $k > 0$, this inequality holds for $0 < k < 6$.

- **Condition 3: Intersection with branches.** For two distinct intersections, the line must also be steeper than the left branch's slope (which is $-k$). Since $3 > -k$ is always true for $k > 0$, the line will always intersect the left branch as long as it is above the vertex. To intersect the right branch, the slope of the line must be less than the slope of the right branch ($3 < k$).

Combining $k > 3$ and $k < 6$:

$$3 < k < 6$$

Final Answers: (a) (i) $y = 10 + k$ (a) (ii) $P = \left(\frac{10}{k}, k\right)$ (b) $x \leq \frac{10-k}{k}$ or $x \geq \frac{10+k}{k}$ (c) $3 < k < 6$

WMA11_P3(IAL)_Winter_2025_Q1

Solution

1. Verification of the root interval

To show that the root α lies in the interval $(0.1, 0.2)$, we apply the **Intermediate Value Theorem**. We evaluate the function $f(x) = 2\sec x + 6x - 3$ at the boundaries of the interval.

- At $x = 0.1$:

$$\begin{aligned} f(0.1) &= 2\sec(0.1) + 6(0.1) - 3 \\ &= \frac{2}{\cos(0.1)} + 0.6 - 3 \\ &\approx 2.010067 + 0.6 - 3 \\ &= -0.389933 \end{aligned}$$

- At $x = 0.2$:

$$\begin{aligned} f(0.2) &= 2\sec(0.2) + 6(0.2) - 3 \\ &= \frac{2}{\cos(0.2)} + 1.2 - 3 \\ &\approx 2.040541 + 1.2 - 3 \\ &= 0.240541 \end{aligned}$$

Since $f(0.1) < 0$ and $f(0.2) > 0$, and the function $f(x)$ is continuous on the interval $[0.1, 0.2]$, there must be at least one root α such that $f(\alpha) = 0$ where $0.1 < \alpha < 0.2$.

2. Derivation of the iterative formula

We rearrange the equation $f(x) = 0$ to express x in terms of itself.

$$\begin{aligned} 2\sec x + 6x - 3 &= 0 \\ 6x &= 3 - 2\sec x \\ x &= \frac{3 - 2\sec x}{6} \\ x &= \frac{3}{6} - \frac{2}{6\cos x} \\ x &= \frac{1}{2} - \frac{1}{3\cos x} \end{aligned}$$

Thus, α is a solution to the equation $x = \frac{1}{2} - \frac{1}{3\cos x}$.

3. Numerical iteration

We use the **Fixed-point iteration** formula $x_{n+1} = \frac{1}{2} - \frac{1}{3\cos x_n}$ starting with $x_1 = 0.15$.

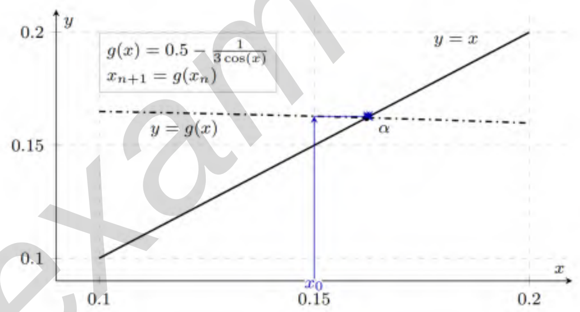
- **(i) Finding x_2:**

$$x_2 = \frac{1}{2} - \frac{1}{3\cos(0.15)}$$
$$\approx 0.5 - \frac{1}{3(0.988771)}$$
$$\approx 0.5 - 0.337119$$
$$= 0.162881$$

Rounding to 4 decimal places, we get $x_2 \approx 0.1629$.

- **(ii) Finding α:** We continue the iteration until the values converge to 4 decimal places.

$$x_3 = \frac{1}{2} - \frac{1}{3\cos(0.162881)} \approx 0.162153$$
$$x_4 = \frac{1}{2} - \frac{1}{3\cos(0.162153)} \approx 0.162195$$
$$x_5 = \frac{1}{2} - \frac{1}{3\cos(0.162195)} \approx 0.162192$$
$$x_6 = \frac{1}{2} - \frac{1}{3\cos(0.162192)} \approx 0.162192$$



The values have stabilized. Rounding to 4 decimal places, we find $\alpha \approx 0.1622$.

(i) $\boxed{x_2 = 0.1629}$

(ii) $\boxed{\alpha = 0.1622}$

WMA11_P3(IAL)_Winter_2025_Q2

Solution

The growth of weed on the surface of a pond is described by the **logarithmic model**:

$$\log_{10} A = 1 + 0.03t$$

where A is the surface area in m^2 and t is the time in weeks.

1. Initial surface area (Part a) The initial state occurs at the start of monitoring, which corresponds to $t = 0$. - Substitute $t = 0$ into the model equation:

$$\log_{10} A = 1 + 0.03(0)$$
$$\log_{10} A = 1$$

- To solve for A, apply the definition of a **logarithm** ($y = \log_b x \Leftrightarrow x = b^y$):

$$A = 10^1$$
$$A = 10$$

The initial surface area is $10\,\mathrm{m}^2$.

2. Time required to reach $25\,\mathrm{m}^2$ (Part b) We are given that after T weeks, the area $A = 25\,\mathrm{m}^2$. - Substitute $A = 25$ and $t = T$ into the equation:

$$\log_{10} 25 = 1 + 0.03T$$

- Rearrange the equation to isolate the term containing T:

$$0.03T = \log_{10} 25 - 1$$

- Solve for T:

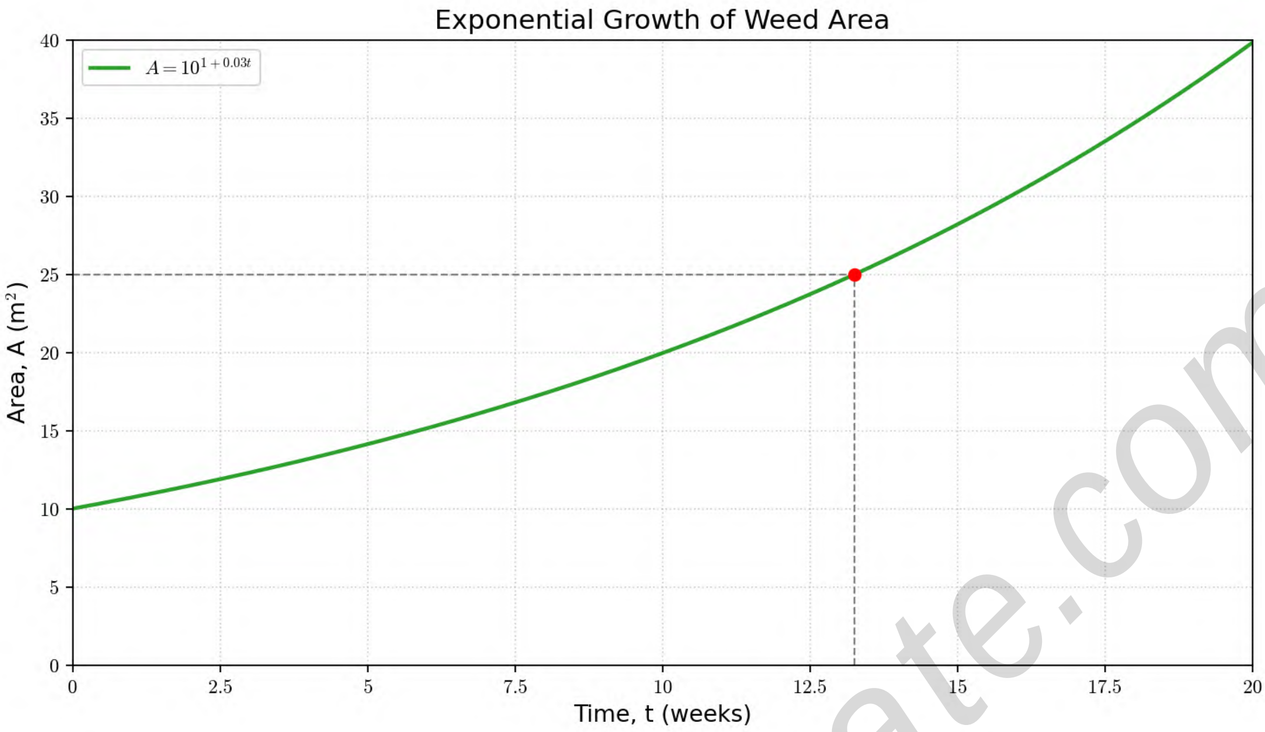
$$T = \frac{\log_{10} 25 - 1}{0.03}$$

- Calculate the numerical value:

$$T \approx \frac{1.397940 - 1}{0.03}$$
$$T \approx \frac{0.397940}{0.03}$$
$$T \approx 13.26466\ldots$$

- Rounding to 2 decimal places as requested:

$$T \approx 13.26$$



Final Answers: (a) $\boxed{10\,\mathrm{m}^2}$ (b) $\boxed{13.26}$

WMA11_P3(IAL)_Winter_2025_Q3

Solution

To determine the range of values for which the function $y = \frac{4x+1}{(x+3)^2}$ is increasing, we must find the interval where its **first derivative** is positive.

1. Differentiation of the function We apply the **quotient rule**, which states that for a function $y = \frac{u}{v}$, the derivative is given by $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$. Let:

- $u = 4x + 1 \implies u' = 4$
- $v = (x+3)^2 \implies v' = 2(x+3)$

Substituting these into the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{4(x+3)^2 - (4x+1) \cdot 2(x+3)}{((x+3)^2)^2} \\ &= \frac{2(x+3)[2(x+3) - (4x+1)]}{(x+3)^4}\end{aligned}$$

2. Simplification of the derivative We simplify the expression by canceling the common factor $(x+3)$ and expanding the terms in the numerator:

$$\begin{aligned}\frac{dy}{dx} &= \frac{2[2x + 6 - 4x - 1]}{(x+3)^3} \\ &= \frac{2(5-2x)}{(x+3)^3}\end{aligned}$$

3. Determining the increasing interval A function is **increasing** when $\frac{dy}{dx} > 0$. Therefore, we solve the inequality:

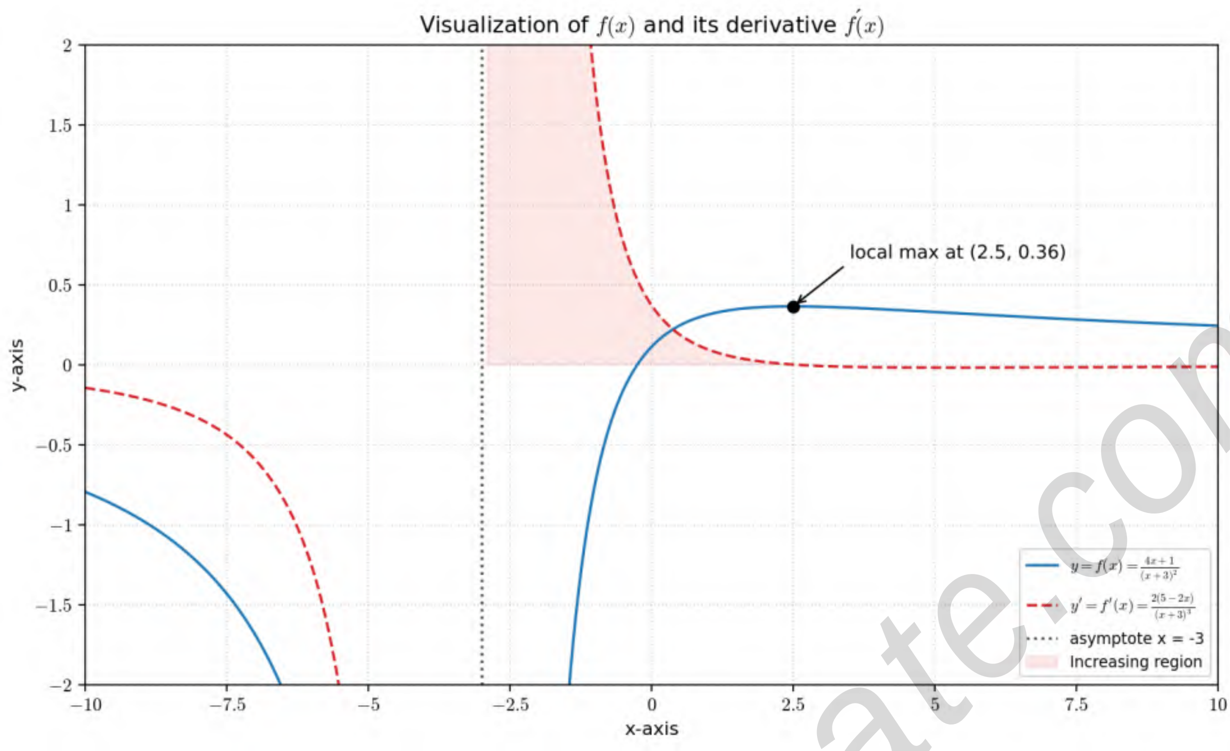
$$\frac{2(5-2x)}{(x+3)^3} > 0$$

To find the critical points, we examine the numerator and the denominator:

- The numerator $2(5-2x) = 0$ when $x = 2.5$.
- The denominator $(x+3)^3 = 0$ when $x = -3$.

We test the sign of $\frac{dy}{dx}$ in the intervals defined by these points:

- For $x < -3$: The numerator is positive ($5 - 2(-4) = 13$) and the denominator is negative ($(-1)^3 = -1$). Thus, $\frac{dy}{dx} < 0$.
- For $-3 < x < 2.5$: The numerator is positive ($5 - 2(0) = 5$) and the denominator is positive ($3^3 = 27$). Thus, $\frac{dy}{dx} > 0$.
- For $x > 2.5$: The numerator is negative ($5 - 2(3) = -1$) and the denominator is positive ($6^3 = 216$). Thus, $\frac{dy}{dx} < 0$.



The function is increasing on the interval where the derivative is strictly positive.

$$\boxed{-3 < x < 2.5}$$

WMA11_P3(IAL)_Winter_2025_Q4

Solution

1. Decomposition of the Rational Function

To find the constants A, B, C, and D, we perform **polynomial long division** on the given expression:

$$\frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4}$$

- **Step 1: Divide the leading terms** The first term of the quotient is $\frac{4x^3}{x^2} = 4x$. Multiply $4x$ by $(x^2 + 4)$:

  $$4x(x^2 + 4) = 4x^3 + 16x$$

  Subtract this from the numerator:

  $$(4x^3 + 2x^2 + 3x + 8) - (4x^3 + 16x) = 2x^2 - 13x + 8$$

- **Step 2: Divide the new leading term** The next term of the quotient is $\frac{2x^2}{x^2} = 2$. Multiply 2 by $(x^2 + 4)$:

  $$2(x^2 + 4) = 2x^2 + 8$$

  Subtract this from the current remainder:

  $$(2x^2 - 13x + 8) - (2x^2 + 8) = -13x$$

Thus, the division yields:

$$\frac{4x^3 + 2x^2 + 3x + 8}{x^2 + 4} = 4x + 2 + \frac{-13x}{x^2 + 4}$$

Comparing this to the form $Ax + B + \frac{Cx+D}{x^2+4}$, we identify:

- **(a) (i)** $A = 4$, $B = 2$, $C = -13$.
- **(a) (ii)** The remainder term is $-13x$, which implies $D = 0$.

2. Evaluation of the Definite Integral

Using the decomposition from part (a), we evaluate the integral:

$$I = \int_1^4 \left(4x + 2 - \frac{13x}{x^2 + 4}\right) dx$$

- **Step 1: Integrate term by term** The integral is split into three parts:

  $$I = \int_1^4 4x\,dx + \int_1^4 2\,dx - \int_1^4 \frac{13x}{x^2 + 4}\,dx$$

- **Step 2: Apply the Power Rule and Logarithmic Integration** For the third term, we use the **substitution method** or the identity $\int \frac{f'(x)}{f(x)} dx = \ln|\ f(x)\ |$. Let $u = x^2 + 4$, then $du = 2x\,dx$, so $x\,dx = \frac{1}{2} du$.

$$\begin{aligned} I &= \left[2x^2 + 2x - \frac{13}{2}\ln(x^2+4)\right]_1^4 \\ &= \left(2(4)^2 + 2(4) - \frac{13}{2}\ln(4^2+4)\right) - \left(2(1)^2 + 2(1) - \frac{13}{2}\ln(1^2+4)\right) \\ &= (32 + 8 - \frac{13}{2}\ln 20) - (2 + 2 - \frac{13}{2}\ln 5) \\ &= 40 - 4 - \frac{13}{2}(\ln 20 - \ln 5) \end{aligned}$$

- **Step 3: Simplify using Logarithm Laws** Using the property $\ln a - \ln b = \ln(\frac{a}{b})$:

$$\begin{aligned} I &= 36 - \frac{13}{2}\ln\left(\frac{20}{5}\right) \\ &= 36 - \frac{13}{2}\ln 4 \\ &= 36 - \frac{13}{2}\ln(2^2) \\ &= 36 - \frac{13}{2}\cdot 2\ln 2 \\ &= 36 - 13\ln 2 \end{aligned}$$

The result is in the form $p + q\ln 2$ where $p = 36$ and $q = -13$.

$\boxed{36 - 13\ln 2}$

WMA11_P3(IAL)_Winter_2025_Q5

Solution

The temperature H of a piece of metal t minutes after being dropped into water is modeled by the **exponential decay** equation:

$$H = 280e^{-0.05t} + 24, \quad t \geq 0$$

1. Initial temperature of the metal The initial temperature occurs at time $t = 0$. Substituting this into the model:

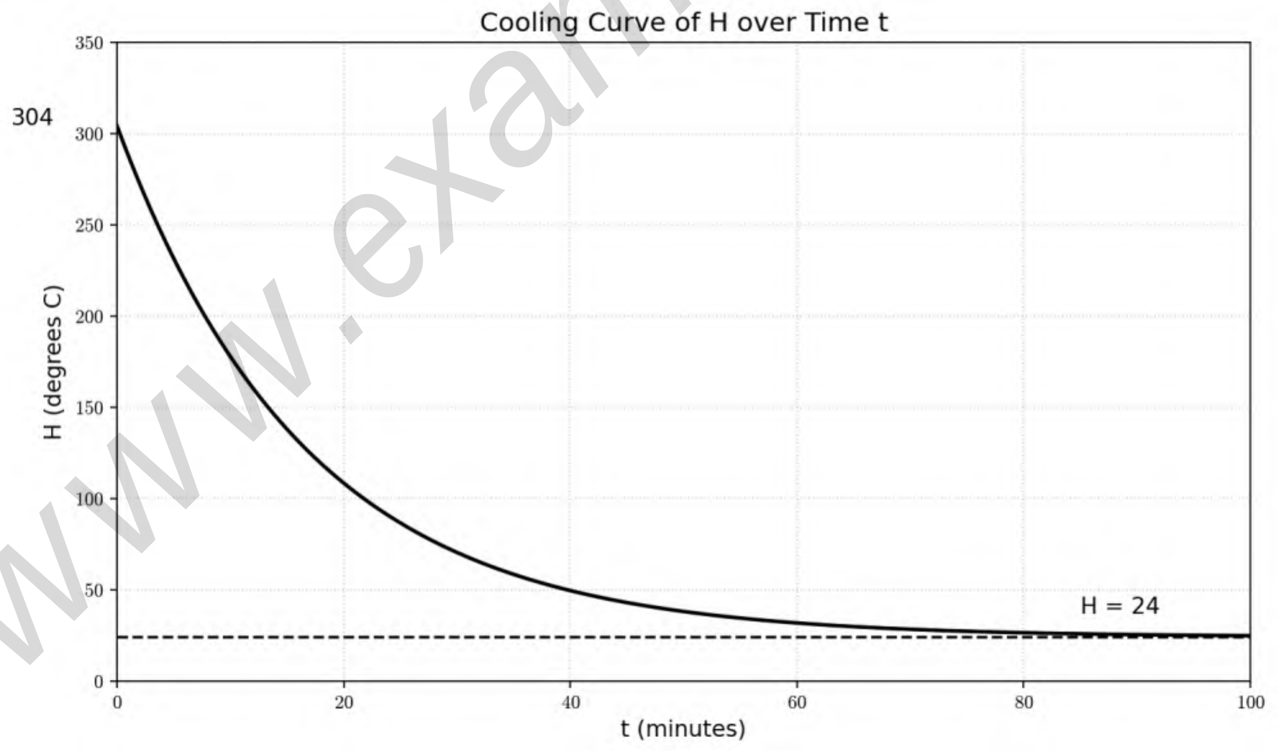
$$\begin{aligned} H(0) &= 280e^{-0.05(0)} + 24 \\ &= 280(1) + 24 \\ &= 304 \end{aligned}$$

The initial temperature is 304° C.

2. Sketch of the graph and asymptote As $t \to \infty$, the term $e^{-0.05t} \to 0$. Therefore, the temperature H approaches a constant value:

$$H \to 24$$

The equation of the **horizontal asymptote** is $H = 24$. The graph starts at $(0, 304)$ and decays exponentially toward this asymptote.



3. Value of t for $H = 144$ To find the time when the temperature reaches 144° C, we solve for t:

$$
\begin{aligned}
144 &= 280e^{-0.05t} + 24 \\
120 &= 280e^{-0.05t} \\
e^{-0.05t} &= \frac{120}{280} = \frac{3}{7} \\
-0.05t &= \ln\left(\frac{3}{7}\right) \\
t &= \frac{\ln(3/7)}{-0.05} \\
t &= -20\ln\left(\frac{3}{7}\right) = 20\ln\left(\frac{7}{3}\right)
\end{aligned}
$$

Using the **natural logarithm** calculation:

$$t \approx 16.945957...$$

Rounding to 2 decimal places, $t = 16.95\,\text{min}$.

4. Differentiation and the differential equation We differentiate H with respect to t using the **chain rule**:

$$
\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt}(280e^{-0.05t} + 24) \\
&= 280(-0.05)e^{-0.05t} \\
&= -14e^{-0.05t}
\end{aligned}
$$

From the original equation, we can express $e^{-0.05t}$ in terms of H:

$$
\begin{aligned}
H - 24 &= 280e^{-0.05t} \\
e^{-0.05t} &= \frac{H - 24}{280}
\end{aligned}
$$

Substitute this back into the expression for dH/dt:

$$
\begin{aligned}
\frac{dH}{dt} &= -14\left(\frac{H - 24}{280}\right) \\
&= -\frac{14}{280}(H - 24) \\
&= -0.05(H - 24) \\
&= 1.2 - 0.05H
\end{aligned}
$$

This matches the form $\frac{dH}{dt} = a + bH$, where:

$$a = 1.2, \quad b = -0.05$$

WMA11_P3(IAL)_Winter_2025_Q6

Solution

1. Finding the inverse function $f^{-1}(x)$

To find the **inverse function**, we set $y = f(x)$ and solve for x in terms of y:

$$y = \frac{4x+3}{x-2}$$

- Multiply both sides by $(x-2)$:

$$y(x-2) = 4x+3$$

- Expand the left side:

$$yx - 2y = 4x + 3$$

- Rearrange the terms to group all x terms on one side:

$$yx - 4x = 2y + 3$$

- Factor out x:

$$x(y-4) = 2y+3$$

- Solve for x:

$$x = \frac{2y+3}{y-4}$$

- Replace x with $f^{-1}(x)$ and y with x:

$$\boxed{f^{-1}(x) = \frac{2x+3}{x-4}, \quad x \neq 4}$$

2. Finding the composite function $ff(x)$

The **composite function** $ff(x)$ is defined as $f(f(x))$:

$$\begin{aligned} ff(x) &= \frac{4\left(\frac{4x+3}{x-2}\right)+3}{\left(\frac{4x+3}{x-2}\right)-2} \\ &= \frac{\frac{4(4x+3)+3(x-2)}{x-2}}{\frac{(4x+3)-2(x-2)}{x-2}} \\ &= \frac{16x+12+3x-6}{4x+3-2x+4} \\ &= \frac{19x+6}{2x+7} \end{aligned}$$

Comparing this to the form $\frac{ax+b}{cx+d}$, we identify the integers:

$$\boxed{a = 19, \quad b = 6, \quad c = 2, \quad d = 7}$$

3. Coordinate transformation

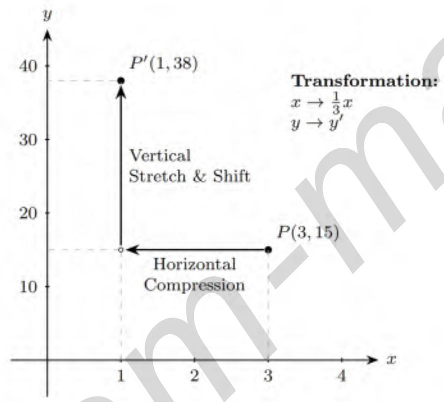
The point $P(3, 15)$ lies on $y = f(x)$. We apply the **function transformations** to the coordinates (x, y) to find the new point P' on the curve $y = 2f(3x) + 8$.

- **Horizontal transformation**: The term $f(3x)$ represents a horizontal stretch by a factor of $1/3$ (or a compression).

$$x' = \frac{x}{3} = \frac{3}{3} = 1$$

- **Vertical transformation**: The term $2f(...) + 8$ represents a vertical stretch by a factor of 2 followed by a vertical translation of $+8$ units.

$$y' = 2y + 8 = 2(15) + 8 = 30 + 8 = 38$$



The coordinates of the mapped point are:

$$\boxed{(1, 38)}$$

WMA11_P3(IAL)_Winter_2025_Q7

Solution

1. Sketching the Absolute Value Graphs

Given that a and b are positive constants with $a > b$:

(i) Graph of $y = |3x - a|$

- **Minimum point:** The minimum value of an **absolute value** function occurs when the expression inside the modulus is zero.

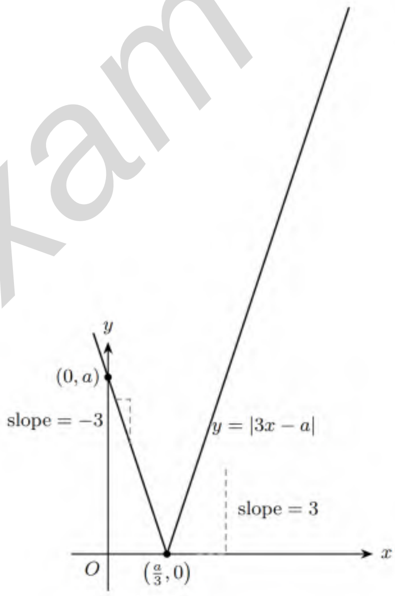
$$3x - a = 0 \Longrightarrow x = \frac{a}{3}$$

  Since $|0| = 0$, the minimum point is $(\frac{a}{3}, 0)$.

- **y-intercept:** Set $x = 0$.

$$y = |3(0) - a| = |-a|$$

  Since $a > 0$, $|-a| = a$. The y-intercept is $(0, a)$.



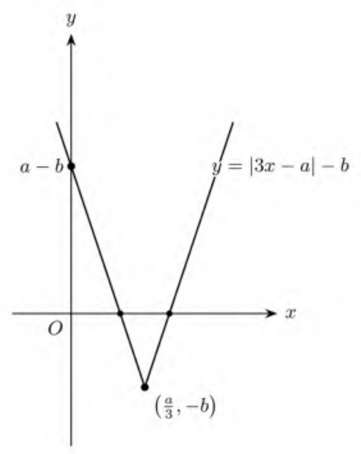
(ii) Graph of $y = |3x - a| - b$

- **Minimum point:** This is a **vertical translation** of the previous graph downward by b units. The minimum occurs at $x = \frac{a}{3}$, giving $y = 0 - b = -b$. The minimum point is $(\frac{a}{3}, -b)$.

- **y-intercept:** Set $x = 0$.

$$y = |3(0) - a| - b = a - b$$

Since $a > b$, $a - b > 0$, so the intercept is above the x-axis. The y-intercept is $(0, a - b)$.



2. Solving the Equation $| 3x - a | - b = 5x$

To solve the equation involving a **modulus**, we consider the two cases for the expression inside the absolute value.

Case 1: $3x - a \geq 0$ (i.e., $x \geq \frac{a}{3}$)

$$(3x - a) - b = 5x$$
$$3x - a - b = 5x$$
$$-a - b = 2x$$
$$x = -\frac{a + b}{2}$$

We must check if this solution satisfies the condition $x \geq \frac{a}{3}$. Since a and b are positive, $-\frac{a+b}{2}$ is negative, while $\frac{a}{3}$ is positive. Thus, this solution is **invalid**.

Case 2: $3x - a < 0$ (i.e., $x < \frac{a}{3}$)

$$-(3x - a) - b = 5x$$
$$-3x + a - b = 5x$$
$$a - b = 8x$$
$$x = \frac{a - b}{8}$$

We must check if this solution satisfies the condition $x < \frac{a}{3}$. We compare $\frac{a-b}{8}$ and $\frac{a}{3}$:

$$\frac{a - b}{8} < \frac{a}{3}$$
$$3(a - b) < 8a$$
$$3a - 3b < 8a$$
$$-3b < 5a$$

Since $a, b > 0$, the inequality $-3b < 5a$ is always true. Therefore, this solution is **valid**.

$$\boxed{x = \frac{a-b}{8}}$$

WMA11_P3(IAL)_Winter_2025_Q8

Solution

1. Solving for θ in $3\csc\theta = 8\cos\theta$

- **Transformation to Sine and Cosine** Using the **reciprocal identity** $\csc\theta = \frac{1}{\sin\theta}$, we rewrite the equation:

$$3\left(\frac{1}{\sin\theta}\right) = 8\cos\theta$$

  Multiplying both sides by $\sin\theta$ (noting $\sin\theta \neq 0$ for $0 < \theta < \pi$):

$$3 = 8\sin\theta\cos\theta$$

- **Application of Double Angle Identity** Recall the **double angle formula** for sine: $\sin 2\theta = 2\sin\theta\cos\theta$. We can rearrange the equation to isolate this term:

$$3 = 4(2\sin\theta\cos\theta)$$
$$3 = 4\sin 2\theta$$
$$\sin 2\theta = \frac{3}{4} = 0.75$$

- **Solving for 2θ** The interval for θ is $0 < \theta < \pi$, so the interval for 2θ is $0 < 2\theta < 2\pi$. The primary value is $2\theta = \arcsin(0.75)$.

$$2\theta_1 \approx 0.84806\,\text{rad}$$
$$2\theta_2 = \pi - 0.84806 \approx 2.29353\,\text{rad}$$

- **Final Values for θ** Dividing by 2 and rounding to 3 significant figures:

$$\theta_1 = \frac{0.84806}{2} \approx 0.424\,\text{rad}$$
$$\theta_2 = \frac{2.29353}{2} \approx 1.15\,\text{rad}$$

$\boxed{\theta = 0.424, 1.15}$

2. Solving for x in $\frac{\tan 2x - \tan 70^\circ}{1 + \tan 2x \tan 70^\circ} = -\frac{3}{8}$

- **Application of Tangent Subtraction Formula** The left-hand side matches the form of the **tangent subtraction identity**: $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$. Let $A = 2x$ and $B = 70^\circ$:

$$\tan(2x - 70^\circ) = -0.375$$

- **Solving for the Angle** The interval for x is $0^\circ < x < 180^\circ$. Thus, the interval for $2x - 70^\circ$ is:

$$2(0^\circ) - 70^\circ < 2x - 70^\circ < 2(180^\circ) - 70^\circ$$
$$-70^\circ < 2x - 70^\circ < 290^\circ$$

  The principal value is $\arctan(-0.375) \approx -20.556^\circ$. Using the periodicity of the tangent function (180°):

$$(2x - 70^\circ)_1 = -20.556^\circ$$
$$(2x - 70^\circ)_2 = -20.556^\circ + 180^\circ = 159.444^\circ$$

(The next value, $159.444^\circ + 180^\circ = 339.444^\circ$, is outside the range).

- **Final Values for x** Solving for x and rounding to one decimal place:

$$2x_1 = -20.556^\circ + 70^\circ = 49.444^\circ \implies x_1 \approx 24.7^\circ$$
$$2x_2 = 159.444^\circ + 70^\circ = 229.444^\circ \implies x_2 \approx 114.7^\circ$$

$\boxed{x = 24.7^\circ, 114.7^\circ}$

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Solution

The problem involves analyzing the function $f(x) = 6\sqrt{x}\ln(4x)$ for $x > 0$. We are asked to find the x-coordinate of the x-intercept P, the exact coordinates of the **stationary point** Q, and the range of a related function $g(x)$.

1. Finding the x-coordinate of P The point P is the x-intercept of the curve $y = f(x)$. At this point, $y = 0$.

$$6\sqrt{x}\ln(4x) = 0$$

Since $x > 0$, $\sqrt{x} \neq 0$. Therefore, we must have:

$$\begin{aligned} \ln(4x) &= 0 \\ 4x &= e^0 \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

The x-coordinate of P is $\boxed{0.25}$.

2. Finding the exact coordinates of Q The point Q is a stationary point, which occurs where the first derivative $f'(x) = 0$. We apply the **product rule** to $f(x) = 6x^{1/2}\ln(4x)$.

- Let $u = 6x^{1/2} \implies \frac{du}{dx} = 3x^{-1/2} = \frac{3}{\sqrt{x}}$
- Let $v = \ln(4x) \implies \frac{dv}{dx} = \frac{1}{4x} \cdot 4 = \frac{1}{x}$

Applying the product rule $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$:

$$\begin{aligned} f'(x) &= 6\sqrt{x}\left(\frac{1}{x}\right) + \ln(4x)\left(\frac{3}{\sqrt{x}}\right) \\ &= \frac{6}{\sqrt{x}} + \frac{3\ln(4x)}{\sqrt{x}} \\ &= \frac{3}{\sqrt{x}}(2 + \ln(4x)) \end{aligned}$$

To find the stationary point, set $f'(x) = 0$:

$$\begin{aligned} 2 + \ln(4x) &= 0 \\ \ln(4x) &= -2 \\ 4x &= e^{-2} \\ x &= \frac{1}{4e^2} \end{aligned}$$

Now, find the corresponding y-coordinate by substituting x back into $f(x)$:

$$\begin{aligned}y &= 6\sqrt{\frac{1}{4e^2}}\ln\left(4 \cdot \frac{1}{4e^2}\right)\\ &= 6\left(\frac{1}{2e}\right)\ln(e^{-2})\\ &= \frac{3}{e}(-2)\\ &= -\frac{6}{e}\end{aligned}$$

The exact coordinates of Q are $\boxed{\left(\frac{1}{4e^2}, -\frac{6}{e}\right)}$.

3. Finding the range of $g(x) = -2f(x)$ From the sketch in Figure 1, the stationary point Q is the **global minimum** of $f(x)$ for $x > 0$.

- The minimum value of $f(x)$ is $y_{\min} = -\frac{6}{e}$.
- As $x \to \infty$, $f(x) \to \infty$.
- As $x \to 0^+$, $f(x) \to 0$ (using **L'Hôpital's rule** or standard limits). Thus, the range of $f(x)$ is $f(x) \geq -\frac{6}{e}$.

The function $g(x)$ is defined as $g(x) = -2f(x)$. This transformation involves a vertical stretch by a factor of 2 and a reflection in the x-axis.

- The minimum value of $f(x)$, which is $-\frac{6}{e}$, becomes the maximum value of $g(x)$:

$$\begin{aligned}g_{\max} &= -2\left(-\frac{6}{e}\right)\\ &= \frac{12}{e}\end{aligned}$$

- Since $f(x)$ can be arbitrarily large, $g(x)$ can be arbitrarily small (negative). Therefore, the range of $g(x)$ is $\boxed{g(x) \leq \frac{12}{e}}$.

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Solution

The problem involves finding the derivative of a curve defined by a trigonometric relation and determining the coordinates of a specific point on that curve based on its gradient.

1. Finding the derivative dx/dy

The equation of the curve is given by:

$$x = 3\cos(2y)$$

To find $\frac{dx}{dy}$, we differentiate x with respect to y using the **chain rule**:

$$\begin{aligned}\frac{dx}{dy} &= \frac{d}{dy}(3\cos(2y)) \\ &= 3 \cdot (-\sin(2y)) \cdot 2 \\ &= -6\sin(2y)\end{aligned}$$

Thus, the derivative is:

$$\boxed{\frac{dx}{dy} = -6\sin(2y)}$$

2. Expressing dy/dx in terms of x

We use the **inverse function theorem** for derivatives, which states that $\frac{dy}{dx} = \frac{1}{dx/dy}$.

$$\frac{dy}{dx} = \frac{1}{-6\sin(2y)}$$

To express this in terms of x, we use the **Pythagorean identity** $\sin^2(2y) + \cos^2(2y) = 1$.

- From the original equation, $\cos(2y) = \frac{x}{3}$.
- Then, $\sin(2y) = \pm\sqrt{1-\cos^2(2y)} = \pm\sqrt{1-\left(\frac{x}{3}\right)^2} = \pm\sqrt{\frac{9-x^2}{9}} = \pm\frac{\sqrt{9-x^2}}{3}$.

In the interval $0 \le y \le \frac{\pi}{2}$, the argument $2y$ ranges from 0 to π. In this range, $\sin(2y) \ge 0$. Therefore, we take the positive root:

$$\sin(2y) = \frac{\sqrt{9-x^2}}{3}$$

Substituting this back into the expression for $\frac{dy}{dx}$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{-6\left(\frac{\sqrt{9-x^2}}{3}\right)} \\ &= \frac{1}{-2\sqrt{9-x^2}} \\ &= \frac{-1/2}{\sqrt{9-x^2}}\end{aligned}$$

Comparing this to the form $\frac{k}{\sqrt{9-x^2}}$, we find $k = -1/2$.

3. Finding the exact values of a and b

The point $P(a, b)$ lies on the curve, so $a = 3\cos(2b)$. The gradient at P is given as $-1/4$. Using the derived formula for the gradient:

$$-\frac{1}{4} = \frac{-1/2}{\sqrt{9-a^2}}$$
$$\sqrt{9-a^2} = \frac{-1/2}{-1/4}$$
$$\sqrt{9-a^2} = 2$$

Squaring both sides:

$$9 - a^2 = 4$$
$$a^2 = 5$$
$$a = \sqrt{5} \quad (\text{since} \quad a > 0)$$

Now, we find b by substituting a into the curve equation:

$$\sqrt{5} = 3\cos(2b)$$
$$\cos(2b) = \frac{\sqrt{5}}{3}$$
$$2b = \arccos\left(\frac{\sqrt{5}}{3}\right)$$
$$b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)$$

Since $a = \sqrt{5} \approx 2.236$ and b is positive, this point corresponds to the location shown in Figure 2.

The exact values are:

$$\boxed{a = \sqrt{5}, \quad b = \frac{1}{2}\arccos\left(\frac{\sqrt{5}}{3}\right)}$$