# WMA11_P2(IAL)_Summer_2019_Q1

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## Solution

### 1. Determining the sequence terms

The sequence $a_1, a_2, a_3, \ldots$ is defined by the **recurrence relation** $a_{n+1} = 4 - a_n$ with the initial term $a_1 = 3$.

- **(i) Find $a_2$** Using the recurrence relation for $n = 1$:

$$\begin{aligned} a_2 &= 4 - a_1 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

$\boxed{a_2 = 1}$

- **(ii) Find $a_{107}$** Let us examine the behavior of the sequence by calculating further terms:

$$\begin{aligned} a_3 &= 4 - a_2 = 4 - 1 = 3 \\ a_4 &= 4 - a_3 = 4 - 3 = 1 \end{aligned}$$

The sequence is **periodic** with a period of 2, oscillating between 3 and 1:

$$a_n = \begin{cases} 3 \text{ if } n \text{ is odd} \\ 1 \text{ if } n \text{ is even} \end{cases}$$

Since $107$ is an odd number:

$$a_{107} = 3$$

$\boxed{a_{107} = 3}$

### 2. Evaluating the summation

We need to find the value of the sum $\sum_{n=1}^{200}(2a_n - 1)$.

- **Step 1: Simplify the general term** Let $b_n = 2a_n - 1$.
  - If $n$ is odd, $a_n = 3$, so $b_n = 2(3) - 1 = 5$.
  - If $n$ is even, $a_n = 1$, so $b_n = 2(1) - 1 = 1$.
- **Step 2: Sum over the range** The sum consists of 200 terms. Since the sequence $b_n$ alternates between 5 and 1, there are exactly 100 terms where $n$ is odd and 100 terms where $n$ is even.

$$\begin{aligned} \sum_{n=1}^{200}(2a_n - 1) &= \sum_{n \in \text{ odd}}^{199} 5 + \sum_{n \in \text{ even}}^{200} 1 \\ &= 100(5) + 100(1) \\ &= 500 + 100 \\ &= 600 \end{aligned}$$

$\boxed{600}$

# WMA11_P2(IAL)_Summer_2019_Q2

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## Solution

### 1. Standard Form of the Circle Equation

To find the center and radius of the circle $C$, we rewrite the given general equation by **completing the square** for both $x$ and $y$ terms. The general equation is:

$$x^2 + y^2 + 4x - 10y - 21 = 0$$

- Group the $x$ and $y$ terms:

$$(x^2 + 4x) + (y^2 - 10y) = 21$$

- Complete the square for $x$: $(x^2 + 4x + 4) - 4 = (x+2)^2 - 4$
- Complete the square for $y$: $(y^2 - 10y + 25) - 25 = (y-5)^2 - 25$
- Substitute these back into the equation:

$$\begin{aligned}(x+2)^2 - 4 + (y-5)^2 - 25 &= 21\\ (x+2)^2 + (y-5)^2 &= 21 + 4 + 25\\ (x+2)^2 + (y-5)^2 &= 50\end{aligned}$$

Comparing this to the standard **circle equation** $(x-h)^2 + (y-k)^2 = r^2$:

- (i) The coordinates of the center $(h, k)$ are $(-2, 5)$.
- (ii) The radius $r$ is $\sqrt{50}$. Simplifying the radical:

$$r = \sqrt{25 \times 2} = 5\sqrt{2}$$

### 2. Equation of the Tangent at Point $P(5, 4)$

The **tangent** to a circle at a point $P$ is perpendicular to the radius connecting the center $O(-2, 5)$ to $P(5, 4)$.

- **Step 1: Find the gradient of the radius $OP$.** The gradient $m_{\text{radius}}$ is given by:

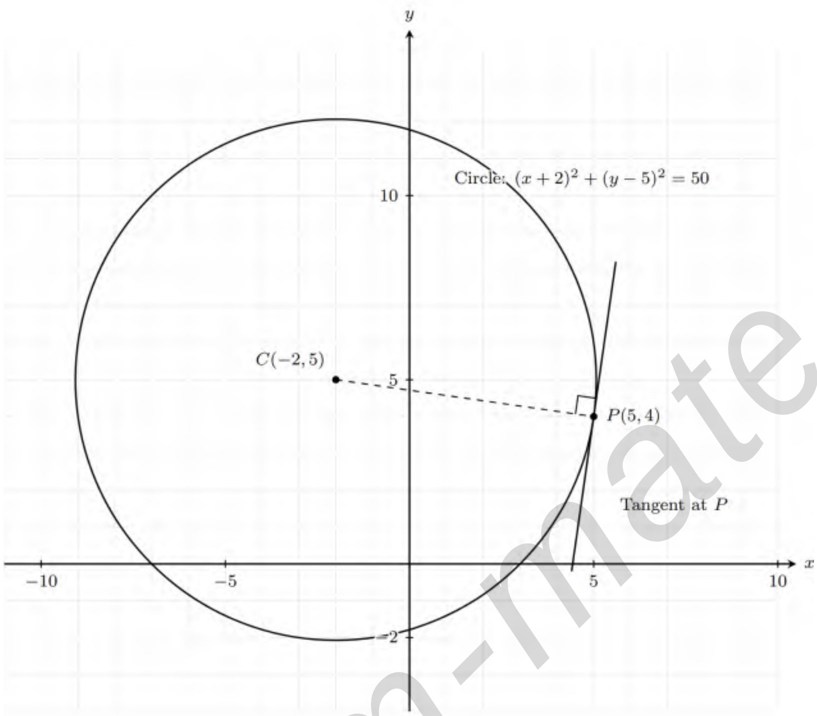
$$\begin{aligned}m_{\text{radius}} &= \frac{y_P - y_O}{x_P - x_O}\\ &= \frac{4-5}{5-(-2)}\\ &= \frac{-1}{7}\end{aligned}$$

- **Step 2: Find the gradient of the tangent.** Since the tangent is perpendicular to the radius, its gradient $m$ satisfies $m \cdot m_{\text{radius}} = -1$:

$$\begin{aligned}m \cdot \left(-\frac{1}{7}\right) &= -1\\ m &= 7\end{aligned}$$

- **Step 3: Determine the equation of the tangent line.** Using the point-slope form $y - y_1 = m(x - x_1)$ with point $P(5, 4)$ and $m = 7$:

$$y - 4 = 7(x - 5)$$
$$y - 4 = 7x - 35$$
$$y = 7x - 31$$




(a) (i) Center: $\boxed{(-2, 5)}$

(b) (ii) Radius: $\boxed{5\sqrt{2}}$

(c) Tangent equation: $\boxed{y = 7x - 31}$

# WMA11_P2(IAL)_Summer_2019_Q3

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## Solution

### 1. Algebraic Proof of the Inequality

To prove that $(x-4)^2 \geq 2x-9$ for all real values of $x$, we begin by expanding the left-hand side and rearranging the inequality into a standard **quadratic** form.

- **Step 1: Expand the left-hand side**

$$(x-4)^2 = x^2 - 8x + 16$$

- **Step 2: Substitute back into the inequality**

$$x^2 - 8x + 16 \geq 2x - 9$$

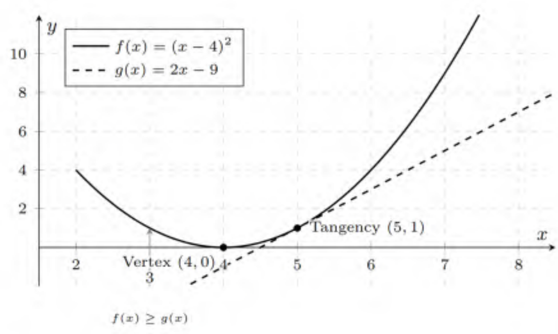
- **Step 3: Rearrange all terms to one side** Subtract $2x$ and add $9$ to both sides:

$$x^2 - 8x - 2x + 16 + 9 \geq 0$$
$$x^2 - 10x + 25 \geq 0$$

- **Step 4: Factor the quadratic expression** The expression $x^2 - 10x + 25$ is a **perfect square trinomial**:

$$(x-5)^2 \geq 0$$

- **Step 5: Conclusion** Since the square of any real number is always non-negative, the statement $(x-5)^2 \geq 0$ is true for all $x \in \mathbb{R}$. Thus, the original inequality $(x-4)^2 \geq 2x-9$ is proven.



### 2. Disproving the Primality Statement

To show that the statement "$2^n + 1$ is a **prime number** for all $n \in \mathbb{N}$" is untrue, we only need to provide a single **counter-example**. We test consecutive values of $n$ (where $n = 1, 2, 3, ...$):

- For $n = 1$: $2^1 + 1 = 3$ (Prime)
- For $n = 2$: $2^2 + 1 = 5$ (Prime)
- For $n = 3$: $2^3 + 1 = 9$

We evaluate the case for $n = 3$:

$$2^3 + 1 = 8 + 1 = 9$$

Since $9 = 3 \times 3$, it is a **composite number** and not prime.

Alternatively, testing $n = 4$:

$$2^4 + 1 = 16 + 1 = 17 \quad \text{(Prime)}$$

Testing $n = 5$:

$$2^5 + 1 = 32 + 1 = 33$$

Since $33 = 3 \times 11$, it is also composite.

**Conclusion:** The existence of $n = 3$ (or $n = 5$) such that $2^n + 1$ is not prime proves that the statement is **untrue**.

$$\boxed{\text{The statement is untrue; counter-example: } n = 3 \implies 2^3 + 1 = 9, \text{ which is composite.}}$$

# WMA11_P2(IAL)_Summer_2019_Q4

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## Solution

### 1. Binomial Expansion of $(2 - \frac{1}{4}x)^6$

To find the first four terms of the expansion, we apply the **Binomial Theorem** for a positive integer power $n$:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

For the expression $(2 - \frac{1}{4}x)^6$, we have $a = 2$, $b = -\frac{1}{4}x$, and $n = 6$. The first four terms correspond to $k = 0, 1, 2, 3$:

- **Term 1 ($k = 0$):**

$$\binom{6}{0}(2)^6\left(-\frac{1}{4}x\right)^0 = 1 \cdot 64 \cdot 1 = 64$$

- **Term 2 ($k = 1$):**

$$\binom{6}{1}(2)^5\left(-\frac{1}{4}x\right)^1 = 6 \cdot 32 \cdot \left(-\frac{1}{4}x\right) = -48x$$

- **Term 3 ($k = 2$):**

$$\binom{6}{2}(2)^4\left(-\frac{1}{4}x\right)^2 = 15 \cdot 16 \cdot \left(\frac{1}{16}x^2\right) = 15x^2$$

- **Term 4 ($k = 3$):**

$$\binom{6}{3}(2)^3\left(-\frac{1}{4}x\right)^3 = 20 \cdot 8 \cdot \left(-\frac{1}{64}x^3\right) = -\frac{160}{64}x^3 = -2.5x^3$$

Combining these, the first four terms in ascending powers of $x$ are:

$$\left(2 - \frac{1}{4}x\right)^6 \approx 64 - 48x + 15x^2 - 2.5x^3$$

### 2. Approximation of the Sum of Binomials

We are asked to find the constants $a$ and $b$ for the expression:

$$\left(2 - \frac{1}{4}x\right)^6 + \left(2 + \frac{1}{4}x\right)^6$$

given that $x^4$ and higher powers are ignored.

- From part (a), we have:

$$\left(2 - \frac{1}{4}x\right)^6 \approx 64 - 48x + 15x^2 - 2.5x^3$$

- Similarly, for $\left(2 + \frac{1}{4}x\right)^6$, the signs of the odd powers of $x$ will flip:

$$\left(2 + \frac{1}{4}x\right)^6 \approx 64 + 48x + 15x^2 + 2.5x^3$$

Summing the two expansions:

$$\begin{aligned}\left(2 - \frac{1}{4}x\right)^6 + \left(2 + \frac{1}{4}x\right)^6 &\approx (64 - 48x + 15x^2 - 2.5x^3) + (64 + 48x + 15x^2 + 2.5x^3) \\ &= (64 + 64) + (-48x + 48x) + (15x^2 + 15x^2) + (-2.5x^3 + 2.5x^3) \\ &= 128 + 30x^2\end{aligned}$$

The linear and cubic terms cancel out due to symmetry. Comparing this to the form $a + bx^2$:

$$a = 128, \quad b = 30$$

**Final Answer:** (a) $\boxed{64 - 48x + 15x^2 - 2.5x^3}$ (b) $\boxed{a = 128, b = 30}$

# WMA11_P2(IAL)_Summer_2019_Q5

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## Solution

The annual profit $P$ (in thousands of pounds) for a company selling watches at a price $x$ (in pounds) is given by the model:

$$P = 12x - x^{3/2} - 120$$

### 1. Finding the maximum possible annual profit

To find the maximum profit, we first determine the **stationary point** by calculating the first derivative of $P$ with respect to $x$ and setting it to zero.

- **Step 1: Differentiate the profit function** Using the **power rule** for differentiation:

$$\begin{aligned}\frac{dP}{dx} &= \frac{d}{dx}(12x) - \frac{d}{dx}(x^{3/2}) - \frac{d}{dx}(120) \\ &= 12 - \frac{3}{2}x^{1/2}\end{aligned}$$

- **Step 2: Solve for the critical value of $x$** Set the first derivative to zero:

$$\begin{aligned}12 - \frac{3}{2}x^{1/2} &= 0 \\ \frac{3}{2}x^{1/2} &= 12 \\ x^{1/2} &= 12 \times \frac{2}{3} \\ x^{1/2} &= 8 \\ x &= 8^2 \\ x &= 64\end{aligned}$$

  The selling price that yields the maximum profit is $£64$.

- **Step 3: Calculate the maximum profit** Substitute $x = 64$ back into the original equation for $P$:

$$\begin{aligned}P &= 12(64) - (64)^{3/2} - 120 \\ &= 768 - (\sqrt{64})^3 - 120 \\ &= 768 - 8^3 - 120 \\ &= 768 - 512 - 120 \\ &= 136\end{aligned}$$

  Since $P$ is measured in thousands of pounds, the maximum profit is $136 \times 1000 = £136,000$.


**2. Justification of the maximum**

To justify that the profit found is a maximum, we use the **second derivative test**.

- **Step 1: Find the second derivative** Differentiate $\frac{dP}{dx} = 12 - \frac{3}{2}x^{1/2}$ with respect to $x$:

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(12 - \frac{3}{2}x^{1/2}\right)$$
$$= 0 - \frac{3}{2} \cdot \frac{1}{2}x^{-1/2}$$
$$= -\frac{3}{4\sqrt{x}}$$

- **Step 2: Evaluate at the critical point** Substitute $x = 64$ into the second derivative:

$$\left.\frac{d^2P}{dx^2}\right|_{x=64} = -\frac{3}{4\sqrt{64}}$$
$$= -\frac{3}{4 \cdot 8}$$
$$= -\frac{3}{32}$$

Since $\frac{d^2P}{dx^2} = -0.09375$, which is less than zero ($< 0$), the **concavity** is downward. This confirms that the stationary point at $x = 64$ is a **local maximum**.

(a) Maximum annual profit: $\boxed{£136,000}$

(b) Since $\frac{d^2P}{dx^2} = -\frac{3}{32} < 0$ at $x = 64$, the profit is a maximum.

# WMA11_P2(IAL)_Summer_2019_Q6

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## Solution

### 1. Determination of the constant $k$

According to the **Factor Theorem**, if $(x-3)$ is a factor of the polynomial $f(x) = kx^3 - 15x^2 - 32x - 12$, then $f(3) = 0$.

- Substitute $x = 3$ into the expression for $f(x)$:

$$\begin{aligned} f(3) &= k(3)^3 - 15(3)^2 - 32(3) - 12 \\ 0 &= 27k - 15(9) - 96 - 12 \\ 0 &= 27k - 135 - 96 - 12 \\ 0 &= 27k - 243 \end{aligned}$$

- Solve for $k$:

$$\begin{aligned} 27k &= 243 \\ k &= \frac{243}{27} \\ k &= 9 \end{aligned}$$

  Thus, it is shown that $k = 9$.

### 2. Full factorization of $f(x)$

With $k = 9$, the polynomial is $f(x) = 9x^3 - 15x^2 - 32x - 12$. Since $(x-3)$ is a factor, we perform **polynomial long division** or **synthetic division** to find the remaining quadratic factor.

- Dividing $9x^3 - 15x^2 - 32x - 12$ by $(x-3)$:

$$9x^3 - 15x^2 - 32x - 12 = (x-3)(9x^2 + 12x + 4)$$

- Now, factorize the quadratic $9x^2 + 12x + 4$. This is a **perfect square trinomial** of the form $(ax+b)^2 = a^2x^2 + 2abx + b^2$:

$$\begin{aligned} 9x^2 + 12x + 4 &= (3x)^2 + 2(3x)(2) + 2^2 \\ &= (3x+2)^2 \end{aligned}$$

- Therefore, the fully factorized form is:

$$\boxed{f(x) = (x-3)(3x+2)^2}$$

### 3. Solving the trigonometric equation

The given equation is $9\cos^3\theta - 15\cos^2\theta - 32\cos\theta - 12 = 0$. Let $x = \cos\theta$. The equation becomes $f(x) = 0$. From the factorization in part (b):

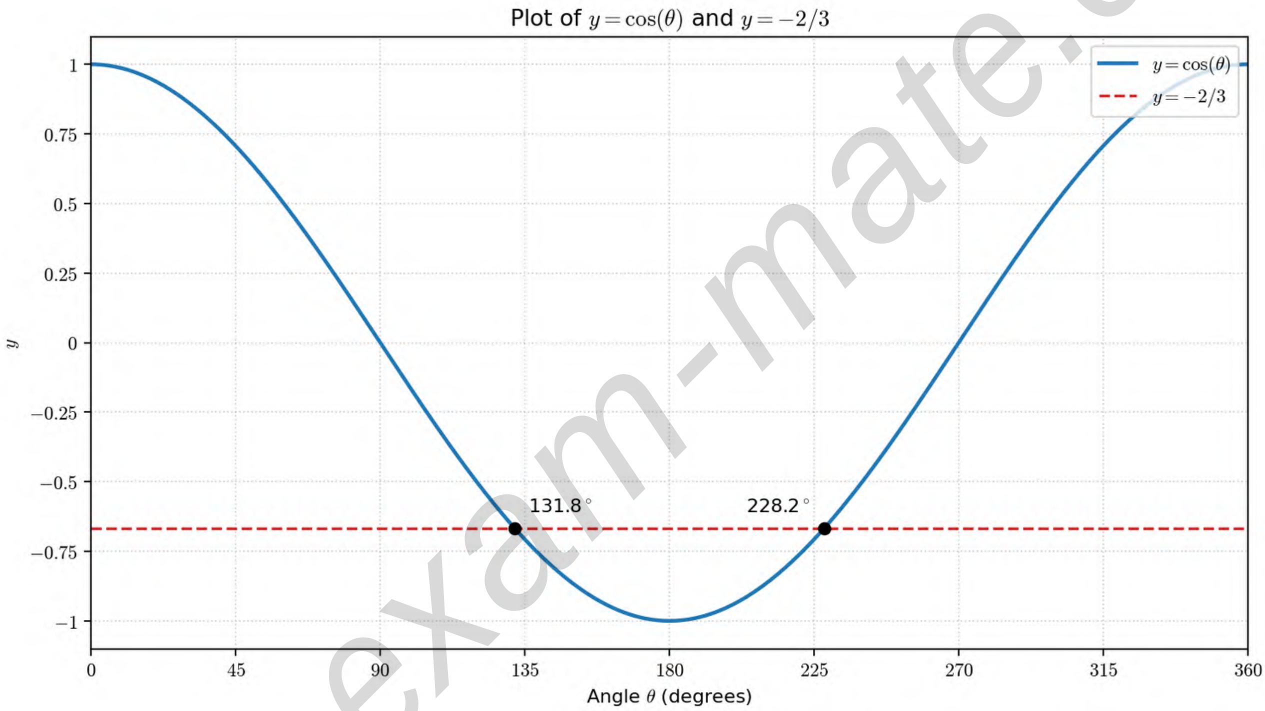
$$(x-3)(3x+2)^2 = 0$$

- This yields two possible values for $x$:
  1. $x - 3 = 0 \implies \cos\theta = 3$
  2. $3x + 2 = 0 \implies \cos\theta = -\frac{2}{3}$

- Analyze the solutions:
  - For $\cos\theta = 3$: Since the range of the **cosine function** is $[-1, 1]$, there are no real solutions for $\theta$.
  - For $\cos\theta = -\frac{2}{3}$:
    - The **principal value** is $\theta = \arccos(-\frac{2}{3})$.
    - Using a calculator: $\theta \approx 131.8103°$.
    - Since cosine is negative in the second and third quadrants, the second solution in the range $0 \leq \theta < 360°$ is:

$$\begin{aligned}\theta_2 &= 360° - 131.8103° \\ &= 228.1897°\end{aligned}$$

- Rounding to one decimal place:
  - $\theta_1 \approx 131.8°$
  - $\theta_2 \approx 228.2°$




$$\boxed{\theta = 131.8°, 228.2°}$$

# WMA11_P2(IAL)_Summer_2019_Q7

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## Solution

### 1. Model A: Arithmetic Progression

Model $A$ assumes a constant annual increase, which corresponds to an **arithmetic progression**. Let $a_n$ be the salary in year $n$.

- Given: $a_1 = 16200$ and $a_{10} = 31500$.
- The general term is $a_n = a_1 + (n-1)d$, where $d$ is the common difference.
  - **Step (a): Determine the salary in year 2** First, calculate the common difference $d$:

$$\begin{aligned} 31500 &= 16200 + (10-1)d \\ 31500 - 16200 &= 9d \\ 15300 &= 9d \\ d &= 1700 \end{aligned}$$

The salary in year 2 is:

$$\begin{aligned} a_2 &= a_1 + d \\ &= 16200 + 1700 \\ &= 17900 \end{aligned}$$

$\boxed{£17900}$

### 2. Model B: Geometric Progression

Model $B$ assumes a constant percentage increase, which corresponds to a **geometric progression**. Let $b_n$ be the salary in year $n$.

- Given: $b_1 = 16200$ and $b_{10} = 31500$.
- The general term is $b_n = b_1 \cdot r^{n-1}$, where $r$ is the common ratio.
  - **Step (b): Determine the salary in year 2** First, calculate the common ratio $r$:

$$\begin{aligned} 31500 &= 16200 \cdot r^{10-1} \\ r^9 &= \frac{31500}{16200} \\ r^9 &= \frac{35}{18} \\ r &= \left(\frac{35}{18}\right)^{1/9} \approx 1.07673... \end{aligned}$$

The salary in year 2 is:

$$\begin{aligned} b_2 &= b_1 \cdot r \\ &= 16200 \cdot \left(\frac{35}{18}\right)^{1/9} \\ &\approx 17443.04... \end{aligned}$$

Rounding to the nearest £10: $\boxed{£17440}$

**3. Comparison of Total Earnings**

- **Step (c): Calculate the difference in total earnings from year 1 to 10** For Model $A$, the total sum $S_A$ is calculated using the **arithmetic series** formula:

$$\begin{aligned} S_A &= \frac{n}{2}(a_1 + a_{10}) \\ &= \frac{10}{2}(16200 + 31500) \\ &= 5 \cdot 47700 \\ &= 238500 \end{aligned}$$

For Model $B$, the total sum $S_B$ is calculated using the **geometric series** formula:

$$\begin{aligned} S_B &= \frac{b_1(r^{10} - 1)}{r - 1} \\ &= \frac{16200\left(\left(\frac{35}{18}\right)^{10/9} - 1\right)}{\left(\frac{35}{18}\right)^{1/9} - 1} \\ &\approx 229719.53... \end{aligned}$$

The difference between the two models is:

$$\begin{aligned} \text{Difference} &= |\ S_A - S_B\ | \\ &= |\ 238500 - 229719.53...\ | \\ &= 8780.46... \end{aligned}$$

Rounding to the nearest £10: $\boxed{£8780}$

# WMA11_P2(IAL)_Summer_2019_Q8

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## Solution

### 1. Solving the Exponential Equation

To find the exact solution of $8^{2x+1} = 6$ in the form $a + b\log_2 3$, we first express the base $8$ as a power of $2$. Since $8 = 2^3$, the equation becomes:

$$(2^3)^{2x+1} = 6$$

Applying the **power of a power rule** $(x^m)^n = x^{mn}$:

$$2^{3(2x+1)} = 6$$

Next, we apply the **logarithm** base $2$ to both sides of the equation:

$$\log_2(2^{3(2x+1)}) = \log_2 6$$
$$3(2x+1) = \log_2(2 \times 3)$$

Using the **product rule of logarithms**, $\log_b(xy) = \log_b x + \log_b y$:

$$6x + 3 = \log_2 2 + \log_2 3$$
$$6x + 3 = 1 + \log_2 3$$

Now, we isolate $x$:

$$6x = 1 - 3 + \log_2 3$$
$$6x = -2 + \log_2 3$$
$$x = \frac{-2}{6} + \frac{1}{6}\log_2 3$$
$$x = -\frac{1}{3} + \frac{1}{6}\log_2 3$$

Comparing this to the required form $a + b\log_2 3$, we identify $a = -1/3$ and $b = 1/6$.

$$\boxed{x = -\frac{1}{3} + \frac{1}{6}\log_2 3}$$

### 2. Solving the Logarithmic Equation

We are given the equation:

$$\log_5(7 - 2y) = 2\log_5(y+1) - 1$$

First, we define the domain constraints for the **logarithmic functions**:

- $7 - 2y > 0 \implies y < 3.5$
- $y + 1 > 0 \implies y > -1$ Thus, the valid range for $y$ is $(-1, 3.5)$.

Using the **power rule of logarithms**, $n\log_b x = \log_b(x^n)$, and expressing $1$ as $\log_5 5$:

$$\log_5(7 - 2y) = \log_5(y+1)^2 - \log_5 5$$

Applying the **quotient rule of logarithms**, $\log_b x - \log_b y = \log_b(x/y)$:

$$\log_5(7-2y)=\log_5\left(\frac{(y+1)^2}{5}\right)$$

Since the logarithm is a **one-to-one function**, we equate the arguments:

$$7-2y=\frac{(y+1)^2}{5}$$

Multiply both sides by $5$ and expand the quadratic:

$$35-10y=y^2+2y+1$$
$$0=y^2+12y-34$$

We solve this **quadratic equation** using the **quadratic formula** $y=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$:

$$\begin{aligned} y &= \frac{-12\pm\sqrt{12^2-4(1)(-34)}}{2(1)} \\ &= \frac{-12\pm\sqrt{144+136}}{2} \\ &= \frac{-12\pm\sqrt{280}}{2} \\ &= \frac{-12\pm 2\sqrt{70}}{2} \\ &= -6\pm\sqrt{70} \end{aligned}$$

We evaluate the two possible solutions:

1. $y_1=-6+\sqrt{70}\approx -6+8.367=2.367$
2. $y_2=-6-\sqrt{70}\approx -6-8.367=-14.367$

Checking against the domain $(-1, 3.5)$, we find that $y_2$ is outside the domain (extraneous solution), while $y_1$ is within the domain.

$\boxed{y=-6+\sqrt{70}}$

# WMA11_P2(IAL)_Summer_2019_Q9

## Solution

### 1. Derivation of the Quadratic Form

To show that the equation $\cos\theta - 1 = 4\sin\theta\tan\theta$ can be rewritten as a quadratic in $\cos\theta$, we utilize the **trigonometric identity** for the tangent function: $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

- Substitute the identity into the original equation:

$$\cos\theta - 1 = 4\sin\theta\left(\frac{\sin\theta}{\cos\theta}\right)$$

- Multiply both sides by $\cos\theta$ (assuming $\cos\theta \neq 0$):

$$\cos^2\theta - \cos\theta = 4\sin^2\theta$$

- Apply the **Pythagorean identity** $\sin^2\theta = 1 - \cos^2\theta$:

$$\cos^2\theta - \cos\theta = 4(1 - \cos^2\theta)$$
$$\cos^2\theta - \cos\theta = 4 - 4\cos^2\theta$$

- Rearrange all terms to one side to form a **quadratic equation**:

$$\cos^2\theta + 4\cos^2\theta - \cos\theta - 4 = 0$$
$$5\cos^2\theta - \cos\theta - 4 = 0$$

### 2. Solving the Equation for $0 \leq x < \frac{\pi}{2}$

The equation $\cos 2x - 1 = 4\sin 2x \tan 2x$ follows the same structure as part (a) with $\theta = 2x$. Thus, we solve:

$$5\cos^2(2x) - \cos(2x) - 4 = 0$$

- Let $u = \cos 2x$. The equation becomes $5u^2 - u - 4 = 0$.
- Factor the quadratic:

$$(5u + 4)(u - 1) = 0$$

- This yields two possible values for $u$:

$$u = 1 \quad \text{or} \quad u = -\frac{4}{5} = -0.8$$

- **Case 1:** $\cos 2x = 1$ Given $0 \leq x < \frac{\pi}{2}$, the range for $2x$ is $0 \leq 2x < \pi$. In this interval, $\cos 2x = 1$ occurs only at:

$$2x = 0$$
$$x = 0$$

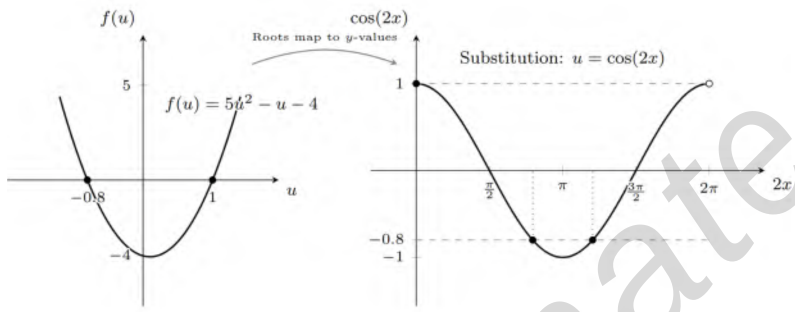
- **Case 2:** $\cos 2x = -0.8$ We find the **principal value** using the inverse cosine function:

$$2x = \arccos(-0.8)$$
$$2x \approx 2.49809... \ \ \text{rad}$$

Since $2.49809... < \pi$, this value is within our range for $2x$. Solving for $x$:

$$x = \frac{2.49809...}{2}$$
$$x \approx 1.24904... \ \text{rad}$$

Rounding to 2 decimal places, we obtain $x = 0.00$ and $x = 1.25$.




$\boxed{x = 0, 1.25}$

# WMA11_P2(IAL)_Summer_2019_Q10

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## Solution

Given the function $f(x) = \frac{36}{x^2} + 2x - 13$ for $x > 0$, we perform the following analysis using **calculus** and properties of **definite integrals**.

### 1. Range of values for which $f(x)$ is increasing

A function is increasing when its first **derivative** is greater than zero ($f'(x) > 0$).

- First, we find $f'(x)$:

$$\begin{aligned} f(x) &= 36x^{-2} + 2x - 13 \\ f'(x) &= -72x^{-3} + 2 \\ &= -\frac{72}{x^3} + 2 \end{aligned}$$

- To find where $f(x)$ is increasing, we solve $f'(x) > 0$:

$$\begin{aligned} -\frac{72}{x^3} + 2 &> 0 \\ 2 &> \frac{72}{x^3} \\ x^3 &> 36 \quad (\text{since} \quad x > 0) \\ x &> \sqrt[3]{36} \end{aligned}$$

The range of values is $\boxed{x > \sqrt[3]{36}}$.

### 2. Evaluation of the definite integral from 2 to 9

To show that $\int_2^9 (\frac{36}{x^2} + 2x - 13)dx = 0$, we find the **antiderivative** $F(x)$:

$$\begin{aligned} \int (36x^{-2} + 2x - 13)dx &= \frac{36x^{-1}}{-1} + \frac{2x^2}{2} - 13x \\ &= -\frac{36}{x} + x^2 - 13x \end{aligned}$$

- Applying the **Fundamental Theorem of Calculus**:

$$\begin{aligned} \left[-\frac{36}{x} + x^2 - 13x\right]_2^9 &= \left(-\frac{36}{9} + 9^2 - 13(9)\right) - \left(-\frac{36}{2} + 2^2 - 13(2)\right) \\ &= (-4 + 81 - 117) - (-18 + 4 - 26) \\ &= (-40) - (-40) \\ &= 0 \end{aligned}$$

Thus, the integral is shown to be $0$.

### 3. Integration properties and constant determination

- **(i) State the value of $\int_6^9 f(x)dx$:** Using the **additivity property** of integrals:

$$\int_2^9 f(x)dx = \int_2^6 f(x)dx + \int_6^9 f(x)dx$$

Given $\int_2^9 f(x)dx = 0$ and $\int_2^6 f(x)dx = -8$:

$$0 = -8 + \int_6^9 f(x)dx$$

$$\int_6^9 f(x)dx = 8$$

The value is $\boxed{8}$.

- **(ii) Find the value of the constant $k$:** We are given $\int_2^6 (\frac{36}{x^2} + 2x + k)dx = 0$. We can split the integral:

$$\int_2^6 \left(\frac{36}{x^2} + 2x - 13 + 13 + k\right)dx = 0$$

$$\int_2^6 f(x)dx + \int_2^6 (k+13)dx = 0$$

$$-8 + \Big[(k+13)x\Big]_2^6 = 0$$

$$-8 + (k+13)(6-2) = 0$$

$$4(k+13) = 8$$

$$k + 13 = 2$$

$$k = -11$$

The value of the constant is $\boxed{k = -11}$.