PURE MATHEMATICS

UNIT P4(IAL)
2020 — 2025

1 - (WMA11/P4(IAL)_Winter_2021_Q4) - *Algebra And Functions*

The curve C is defined by the parametric equations

$$x = \frac{1}{t} + 2 \qquad y = \frac{1-2t}{3+t} \qquad t > 0$$

(a) Show that the equation of C can be written in the form $y = \mathrm{g}(x)$ where g is the function

$$\mathrm{g}(x) = \frac{ax+b}{cx+d} \qquad x > k$$

where a, b, c, d and k are integers to be found.

(5)

(b) Hence, or otherwise, state the range of g.

(2)

2 - (WMA11/P4(IAL)_Summer_2024_Q5) - *Coordinate Geometry In The (x, Y) Plane, Integration, Algebra And Functions*

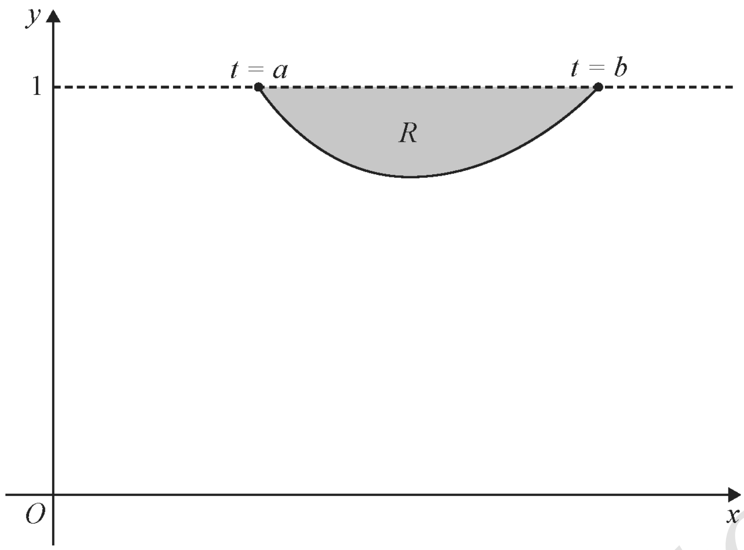

Figure 2

Figure 2 shows a sketch of the curve defined by the parametric equations

$$x = t^2 + 2t \qquad y = \frac{2}{t(3-t)} \qquad a \leqslant t \leqslant b$$

where a and b are constants.

The ends of the curve lie on the line with equation $y = 1$

(a) Find the value of a and the value of b

(2)

The region R, shown shaded in Figure 2, is bounded by the curve and the line with equation $y = 1$

(b) Show that the area of region R is given by

$$M - k\int_a^b \frac{t+1}{t(3-t)}\,\mathrm{d}t$$

where M and k are constants to be found.

(5)

(c) (i) Write $\frac{t+1}{t(3-t)}$ in partial fractions.

(ii) Use algebraic integration to find the exact area of R, giving your answer in simplest form.

(6)

3 - (WMA11/P4(IAL)_Summer_2024_Q6) - *Vectors, Algebra And Functions*

With respect to a fixed origin O, the line l_1 is given by the equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(8\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

where λ is a scalar parameter.

The point A lies on l_1

Given that $|\overrightarrow{OA}| = 5\sqrt{10}$

(a) show that at A the parameter λ satisfies

$$81\lambda^2 + 52\lambda - 220 = 0$$

(3)

Hence

(b) (i) show that one possible position vector for A is $-15\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$

(ii) find the other possible position vector for A.

(3)

The line l_2 is parallel to l_1 and passes through O.

Given that

- $\overrightarrow{OA} = -15\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$
- point B lies on l_2 where $|\overrightarrow{OB}| = 4\sqrt{10}$

(c) find the area of triangle OAB, giving your answer to one decimal place.

(4)

4 - (WMA11/P4(IAL)_Summer_2024_Q9) - *Integration, Trigonometry, Algebra And Functions*

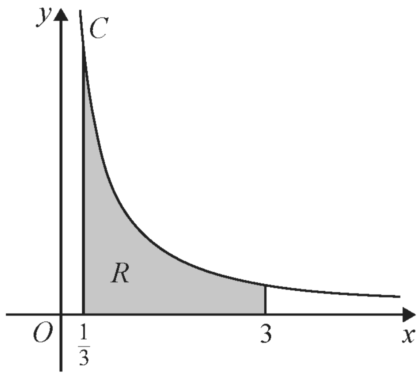

Figure 3

The curve C, shown in Figure 3, has equation

$$y = \frac{x^{-\frac{1}{4}}}{\sqrt{1+x}\left(\arctan\sqrt{x}\right)}$$

The region R, shown shaded in Figure 3, is bounded by C, the line with equation $x = 3$, the x-axis and the line with equation $x = \frac{1}{3}$

The region R is rotated through 360° about the x-axis to form a solid.

Using the substitution $\tan u = \sqrt{x}$

(a) show that the volume V of the solid formed is given by

$$k\int_a^b \frac{1}{u^2}\,\mathrm{d}u$$

where k, a and b are constants to be found.

(6)

(b) Hence, using algebraic integration, find the value of V in simplest form.

(3)

ANSWERS

1 - (WMA11/P4(IAL)_Winter_2021_Q4) - *Algebra And Functions*

(a)	$k=2$ or $x>2$	B1
	$t=\frac{1}{x-2}\Rightarrow y=\frac{1-\frac{2}{x-2}}{3+\frac{1}{x-2}}$	M1 A1
	$\frac{1-\frac{2}{x-2}}{3+\frac{1}{x-2}}=\frac{x-2-2}{\dots}$ or $\frac{\dots}{3(x-2)+1}$	A1 (M1 on EPEN)
	$y=\frac{x-4}{3x-5}$	A1
		(5)
(b)	$-2<\mathrm{g}<\frac{1}{3}$	M1 A1
		(2)
		(7 marks)

2 - (WMA11/P4(IAL)_Summer_2024_Q5) - *Coordinate Geometry In The (x, Y) Plane, Integration, Algebra And Functions*

(a)	$y=\frac{2}{t(3-t)}=1 \Rightarrow 2=3t-t^2 \Rightarrow t=\ldots$	**M1**
	$\left(t^2-3t+2=(t-1)(t-2)=0 \Rightarrow\right) a=1, b=2$	**A1**
		(2)
(b)	Area under curve $=\int_{t="1"}^{t="2"} y\,\mathrm{d}x=\int_1^2 y\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t=\int_1^2 \frac{2}{t(3-t)}\times(2t+2)\,\mathrm{d}t$	**M1** **A1**
	$t=1 \Rightarrow x=3, \quad t=2 \Rightarrow x=8$	**B1**
	So area of $R = 1\times(8-3)-\int_1^2 \frac{2}{t(3-t)}\times(2t+2)\,\mathrm{d}t$	**M1**
	$=5-4\int_1^2 \frac{t+1}{t(3-t)}\mathrm{d}t$	**A1**
		(5)
(c)(i)	$\frac{t+1}{t(3-t)}=\frac{A}{t}+\frac{B}{3-t}$	**M1**
	$\Rightarrow t+1=A(3-t)+Bt$ E.g. $t=0 \Rightarrow 1=3A \Rightarrow A=\frac{1}{3}$; $t=3 \Rightarrow 4=3B \Rightarrow B=\frac{4}{3}$	**M1**
	$\frac{t+1}{t(3-t)}=\frac{1}{3t}+\frac{4}{3(3-t)}$	**A1**
(ii)	$\int \frac{t+1}{t(3-t)}\mathrm{d}t=\int \frac{1}{3t}+\frac{4}{3(3-t)}\mathrm{d}t=\frac{1}{3}\ln t-\frac{4}{3}\ln(3-t)$	**M1**
	$\text{Area}=5-4\left[\frac{1}{3}\ln t-\frac{4}{3}\ln(3-t)\right]_1^2=5-4\left(\frac{1}{3}\ln 2-\frac{4}{3}\ln 1-\frac{1}{3}\ln 1+\frac{4}{3}\ln 2\right)$ $=5-4\left(\frac{1}{3}\ln 2+\frac{4}{3}\ln 2\right)$	**dM1**
	$=5-\frac{20}{3}\ln 2$	**A1**
		(6)
		(13 marks)

3 - (WMA11/P4(IAL)_Summer_2024_Q6) - *Vectors, Algebra And Functions*

(a)	$\lvert\overrightarrow{OA}\rvert^2 = (1+8\lambda)^2 + (2-\lambda)^2 + (5+4\lambda)^2$	**M1**
	$\lvert\overrightarrow{OA}\rvert = 5\sqrt{10} \Rightarrow (1+8\lambda)^2 + (2-\lambda)^2 + (5+4\lambda)^2 = 250$	**M1**
	$\Rightarrow 64\lambda^2 + 16\lambda + 1 + \lambda^2 - 4\lambda + 4 + 16\lambda^2 + 40\lambda + 25 = 250$ $\Rightarrow 81\lambda^2 + 52\lambda - 220 = 0\,*$	**A1***
		(3)
(b)	$81\lambda^2 + 52\lambda - 220 = 0 \Rightarrow (81\lambda - 110)(\lambda + 2) = 0 \Rightarrow \lambda = ...\left(-2, \frac{110}{81}\right)$ $\Rightarrow \overrightarrow{OA} = \begin{pmatrix} 1+8\times\text{"their}\lambda\text{"} \\ 2-\text{"their}\lambda\text{"} \\ 5+4\times\text{"their}\lambda\text{"} \end{pmatrix}$	**M1**
	E.g $\Rightarrow \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} - 2\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -15 \\ 4 \\ -3 \end{pmatrix}\,*$; or $\overrightarrow{OA} = \begin{pmatrix} 1+8\times\frac{110}{81} \\ 2-\frac{110}{81} \\ 5+4\times\frac{110}{81} \end{pmatrix} = \begin{pmatrix} \frac{961}{81} \\ \frac{52}{81} \\ \frac{845}{81} \end{pmatrix}$	**A1*; A1**
		(3)
(c)	$\cos\theta = \pm\dfrac{\begin{pmatrix} -15 \\ 4 \\ -3 \end{pmatrix}.\begin{pmatrix} 8 \\ -1 \\ 4 \end{pmatrix}}{\sqrt{15^2+4^2+3^2}\sqrt{8^2+1^2+4^2}} = ...$	**M1**
	$= \pm\dfrac{-136}{45\sqrt{10}}\ (= \pm 0.9557...)$ or e.g. $\theta =$ awrt $0.299(rad)/17.1°$	**A1**
	Area $OAB = \frac{1}{2}5\sqrt{10}\times 4\sqrt{10}\sin\theta = ...$	**M1**
	= awrt 29.4	**A1**
		(4)
		(10 marks)

4 - (WMA11/P4(IAL)_Summer_2024_Q9) - *Integration, Trigonometry, Algebra And Functions*

(a)	Volume, $V = \pi\int_{\frac{1}{3}}^{3} y^2\,dx = \pi\int_{\frac{1}{3}}^{3} \frac{x^{-\frac{1}{2}}}{(1+x)\left(\arctan\left(\sqrt{x}\right)\right)^2}\,dx$	**B1**
	$\tan u = \sqrt{x} \Rightarrow \sec^2 u\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ oe	**M1** **A1**
	$\Rightarrow V = (\pi)\int \frac{2\times\frac{1}{2}x^{-\frac{1}{2}}}{\left(\arctan\left(\sqrt{x}\right)\right)^2(1+x)}\,dx = (\pi)\int\frac{2}{u^2\left(1+\tan^2 u\right)}\sec^2 u\,du$	**dM1**
	$= \pi\int\frac{2}{u^2\cancel{\sec^2 u}}\cancel{\sec^2 u}\,du = 2\pi\int\frac{1}{u^2}\,du$	**A1**
	$x = 3 \Rightarrow u = \arctan\sqrt{3} = \frac{\pi}{3}$, $x = \frac{1}{3} \Rightarrow u = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ $\Rightarrow V = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{u^2}\,du$	**B1**
		(6)
(b)	$V = (2\pi)\left[-\frac{1}{u}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$	**M1**
	$= (2\pi)\left[-\frac{1}{\pi/3} - \left(-\frac{1}{\pi/6}\right)\right]$ Or $= (2\pi)\left[-\frac{1}{\arctan\sqrt{x}}\right]_{\frac{1}{3}}^{3} = (2\pi)\left(-\frac{1}{\arctan\sqrt{3}} - \left(-\frac{1}{\arctan\sqrt{1/3}}\right)\right)$	**dM1**
	$= 2\pi\left(-\frac{3}{\pi}+\frac{6}{\pi}\right) = 6$	**A1**
		(3)
		(9 marks)