9231_21_Summer_2020_Q1

Solution

The given problem is a **first-order linear ordinary differential equation** of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x) = 5$ and $Q(x) = e^{-7x}$. We are given the **initial condition** $y(0) = 0$.

1. Determine the integrating factor To solve this equation, we first find the **integrating factor**, denoted by $\mu(x)$:

$$\begin{aligned}\mu(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 5\,dx} \\ &= e^{5x}\end{aligned}$$

2. Multiply the differential equation by the integrating factor Multiplying both sides of the original equation by $\mu(x) = e^{5x}$ yields:

$$e^{5x}\frac{dy}{dx} + 5e^{5x}y = e^{5x} \cdot e^{-7x}$$

The left-hand side is the derivative of the product $y \cdot e^{5x}$ by the **product rule**:

$$\frac{d}{dx}(ye^{5x}) = e^{-2x}$$

3. Integrate both sides Integrate with respect to x:

$$ye^{5x} = \int e^{-2x}\,dx$$

$$ye^{5x} = -\frac{1}{2}e^{-2x} + C$$

where C is the constant of integration.

4. Apply the initial condition We are given that $y = 0$ when $x = 0$. Substituting these values into the equation:

$$(0)e^{5(0)} = -\frac{1}{2}e^{-2(0)} + C$$

$$0 = -\frac{1}{2}(1) + C$$

$$C = \frac{1}{2}$$

5. Solve for y Substitute $C = \frac{1}{2}$ back into the general solution:

$$ye^{5x} = -\frac{1}{2}e^{-2x} + \frac{1}{2}$$

To isolate y, multiply the entire equation by e^{-5x}:

$$y = e^{-5x}\left(-\frac{1}{2}e^{-2x} + \frac{1}{2}\right)$$

$$y = -\frac{1}{2}e^{-7x} + \frac{1}{2}e^{-5x}$$

Factoring out $\frac{1}{2}$ gives the final form:

$$y = \frac{1}{2}(e^{-5x} - e^{-7x})$$

$$\boxed{y = \frac{1}{2}(e^{-5x} - e^{-7x})}$$

9231_21_Summer_2020_Q2

Solution

1. Logarithmic Differentiation

To show the derivative of $y = 2^x$ using **logarithmic differentiation**, we first take the natural logarithm of both sides:

$$\ln y = \ln(2^x)$$
$$\ln y = x \ln 2$$

Differentiating both sides with respect to x using the **chain rule** on the left-hand side:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln 2)$$
$$\frac{1}{y}\frac{dy}{dx} = \ln 2$$
$$\frac{dy}{dx} = y \ln 2$$

Substituting $y = 2^x$ back into the equation:

$$\boxed{\frac{dy}{dx} = 2^x \ln 2}$$

2. Second Derivative

To find the second derivative, we differentiate the first derivative $\frac{dy}{dx} = (\ln 2)2^x$ with respect to x. Since $\ln 2$ is a constant:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}((\ln 2)2^x)$$
$$= (\ln 2)\frac{d}{dx}(2^x)$$

Using the result from part (a), where $\frac{d}{dx}(2^x) = 2^x \ln 2$:

$$\frac{d^2y}{dx^2} = (\ln 2)(2^x \ln 2)$$
$$= 2^x(\ln 2)^2$$

$$\boxed{\frac{d^2y}{dx^2} = 2^x(\ln 2)^2}$$

3. Maclaurin Series Expansion

The **Maclaurin series** for a function $f(x)$ is given by:

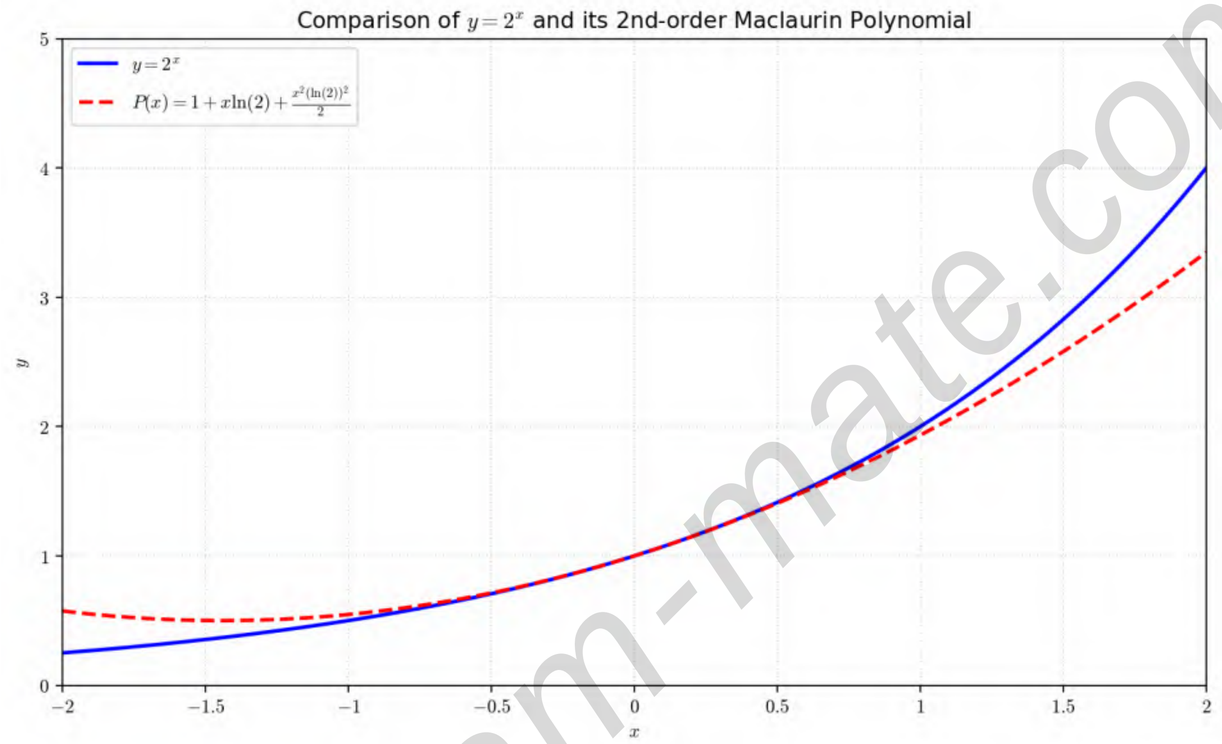
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ...$$

Let $f(x) = 2^x$. We evaluate the function and its derivatives at $x = 0$:

- $f(0) = 2^0 = 1$
- $f'(0) = 2^0 \ln 2 = \ln 2$
- $f''(0) = 2^0 (\ln 2)^2 = (\ln 2)^2$

Substituting these values into the series formula for the first three terms:

$$2^x \approx 1 + (\ln 2)x + \frac{(\ln 2)^2}{2}x^2$$



$$\boxed{2^x \approx 1 + x \ln 2 + \frac{x^2 (\ln 2)^2}{2}}$$

9231_21_Summer_2020_Q3

Solution

1. Finding the roots of the equation $z^3 = -1 - i$

To find the roots, we first express the complex number $-1 - i$ in **exponential form** $re^{i\theta}$.

- The modulus r is:

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

- The argument θ is found in the third quadrant (since both real and imaginary parts are negative):

$$\theta = \pi + \arctan\left(\frac{-1}{-1}\right) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

  Thus, the equation is:

$$z^3 = \sqrt{2}e^{i\left(\frac{5\pi}{4}+2n\pi\right)}, \quad n \in \mathbb{Z}$$

Applying **De Moivre's Theorem** to find the cube roots $z = (z^3)^{1/3}$:

$$z = (\sqrt{2})^{1/3}e^{i\left(\frac{5\pi/4+2n\pi}{3}\right)} = 2^{1/6}e^{i\left(\frac{5\pi}{12}+\frac{2n\pi}{3}\right)}$$

For $n = 0, 1, 2$, we obtain the three roots in the range $0 \le \theta < 2\pi$:

- For $n = 0$: $z_1 = 2^{1/6}e^{i\frac{5\pi}{12}}$
- For $n = 1$: $z_2 = 2^{1/6}e^{i\left(\frac{5\pi}{12}+\frac{8\pi}{12}\right)} = 2^{1/6}e^{i\frac{13\pi}{12}}$
- For $n = 2$: $z_3 = 2^{1/6}e^{i\left(\frac{5\pi}{12}+\frac{16\pi}{12}\right)} = 2^{1/6}e^{i\frac{21\pi}{12}} = 2^{1/6}e^{i\frac{7\pi}{4}}$

$\boxed{z_1 = 2^{1/6}e^{i\frac{5\pi}{12}}, z_2 = 2^{1/6}e^{i\frac{13\pi}{12}}, z_3 = 2^{1/6}e^{i\frac{7\pi}{4}}}$

2. Expressing $w = z_1^{3k} + z_2^{3k} + z_3^{3k}$ in the form $Re^{i\alpha}$

Since z_1, z_2, z_3 are all roots of the equation $z^3 = -1 - i$, it follows that:

$$z_1^3 = z_2^3 = z_3^3 = -1 - i$$

We can substitute this into the expression for w:

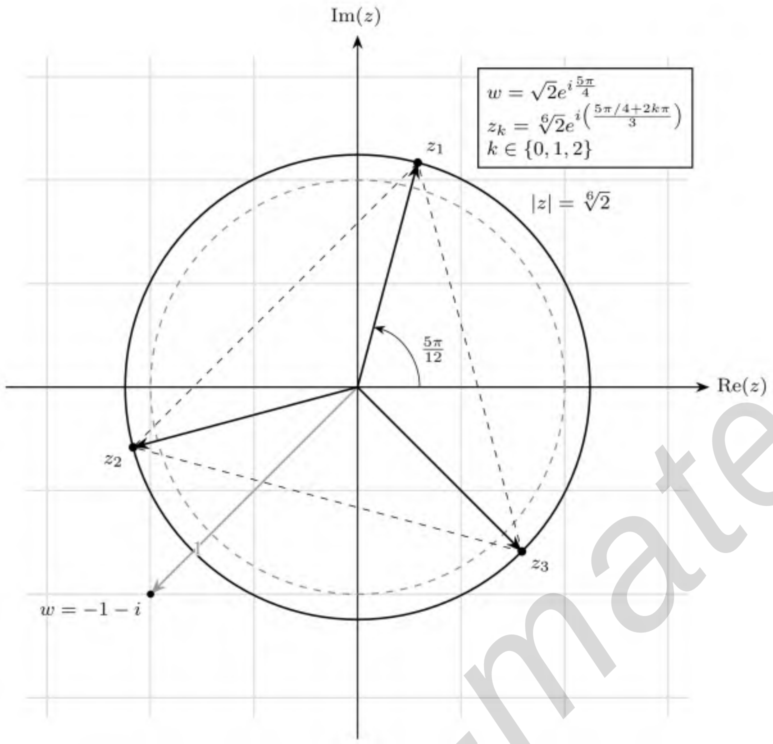
$$\begin{aligned} w &= (z_1^3)^k + (z_2^3)^k + (z_3^3)^k \\ &= (-1-i)^k + (-1-i)^k + (-1-i)^k \\ &= 3(-1-i)^k \end{aligned}$$

Using the exponential form of $-1 - i$ derived in part (a), which is $\sqrt{2}e^{i\frac{5\pi}{4}}$:

$$\begin{aligned} w &= 3\left(\sqrt{2}e^{i\frac{5\pi}{4}}\right)^k \\ &= 3(2^{1/2})^k e^{i\frac{5k\pi}{4}} \\ &= 3 \cdot 2^{k/2}e^{i\frac{5k\pi}{4}} \end{aligned}$$

The expression is now in the form $Re^{i\alpha}$ where:

- $R = 3 \cdot 2^{k/2}$
- $\alpha = \frac{5k\pi}{4}$



$$\boxed{R = 3 \cdot 2^{k/2}, \alpha = \frac{5k\pi}{4}}$$

9231_21_Summer_2020_Q4

Solution

The problem asks us to estimate the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ using a **Riemann sum** approach with n rectangles of equal width.

1. Upper Bound Estimation

- The interval $[0, 1]$ is divided into n sub-intervals, each of width $\Delta x = \frac{1}{n}$.
- The diagram shows rectangles whose heights are determined by the value of the function at the right-hand endpoint of each sub-interval. Since $f(x) = x^2$ is an increasing function on $[0, 1]$, this **Right Riemann Sum** (R_n) provides an upper bound for the integral.
- The k-th rectangle has a height $f(x_k) = \left(\frac{k}{n}\right)^2$ for $k = 1, 2, ..., n$.
- The sum of the areas of these n rectangles is:

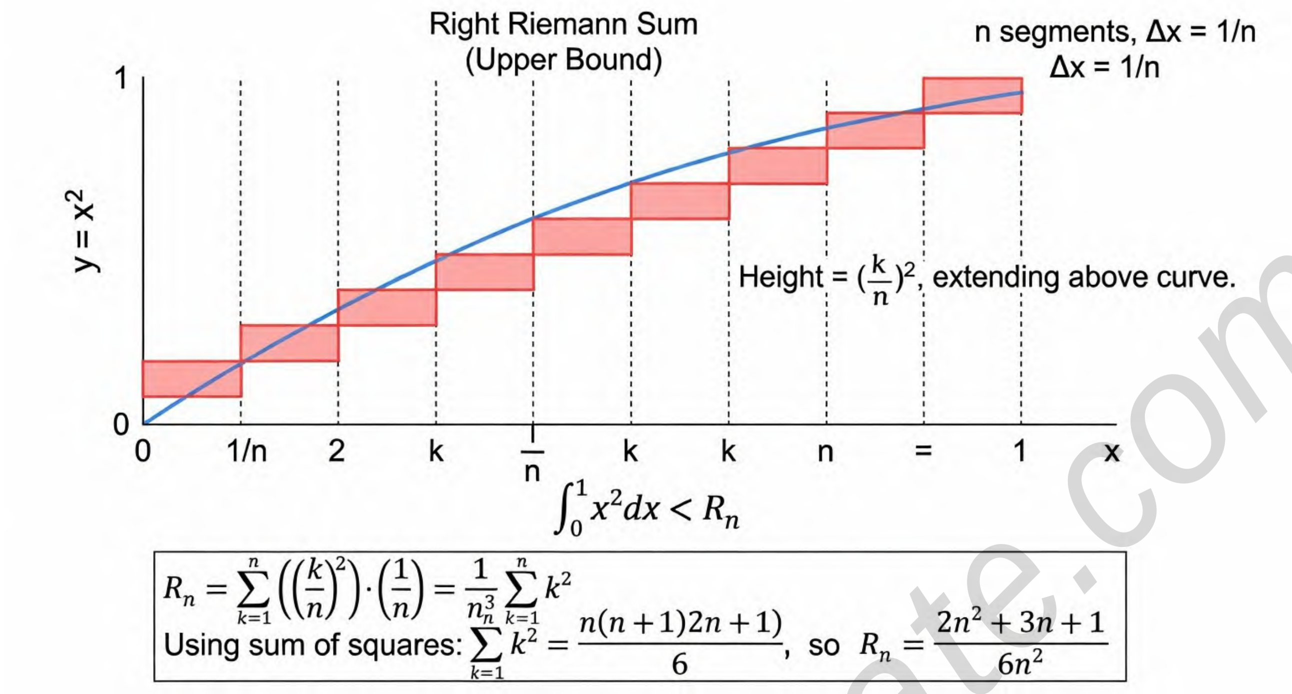
$$\begin{aligned} R_n &= \sum_{k=1}^{n} f(x_k)\Delta x \\ &= \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3} \sum_{k=1}^{n} k^2 \end{aligned}$$

- Using the standard formula for the **sum of squares**, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$:

$$\begin{aligned} R_n &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6n^2} \\ &= \frac{2n^2 + 3n + 1}{6n^2} \end{aligned}$$

- Since $f(x) = x^2$ is strictly increasing, the area under the curve is strictly less than the sum of the right-endpoint rectangles:

$$\int_0^1 x^2 dx < \frac{2n^2 + 3n + 1}{6n^2}$$



2. Lower Bound Estimation

- To find a lower bound, we use the **Left Riemann Sum** (L_n), where the height of each rectangle is determined by the function value at the left-hand endpoint of each sub-interval.
- The k-th rectangle has a height $f(x_{k-1}) = \left(\frac{k-1}{n}\right)^2$ for $k = 1, 2, ..., n$.
- The sum of the areas is:

$$\begin{aligned} L_n &= \sum_{k=1}^{n} f(x_{k-1})\Delta x \\ &= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n} \\ &= \frac{1}{n^3}\sum_{j=0}^{n-1} j^2 \end{aligned}$$

- Applying the sum of squares formula for $n-1$ terms:

$$\begin{aligned} L_n &= \frac{1}{n^3} \cdot \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} \\ &= \frac{1}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6} \\ &= \frac{(n-1)(2n-1)}{6n^2} \\ &= \frac{2n^2-3n+1}{6n^2} \end{aligned}$$

- Since $f(x) = x^2$ is increasing, the left-endpoint rectangles lie entirely below the curve (except at the endpoints), providing a lower bound:

$$\int_0^1 x^2 dx > \frac{2n^2-3n+1}{6n^2}$$

$$\boxed{\frac{2n^2-3n+1}{6n^2}}$$

9231_21_Summer_2020_Q5

Solution

1. Finding the intersection point a

To find the intersection point of the curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$, we set the equations equal to each other:

$$\cosh x = \sinh 2x$$

Using the **double-angle identity** for hyperbolic sine, $\sinh 2x = 2\sinh x \cosh x$, we have:

$$\cosh x = 2\sinh x \cosh x$$

Rearranging the equation:

$$\cosh x(1 - 2\sinh x) = 0$$

Since the **hyperbolic cosine** function $\cosh x = \frac{e^x + e^{-x}}{2}$ is always greater than or equal to 1 for all real x, $\cosh x \neq 0$. Thus, we must have:

$$1 - 2\sinh x = 0 \Longrightarrow \sinh x = \frac{1}{2}$$

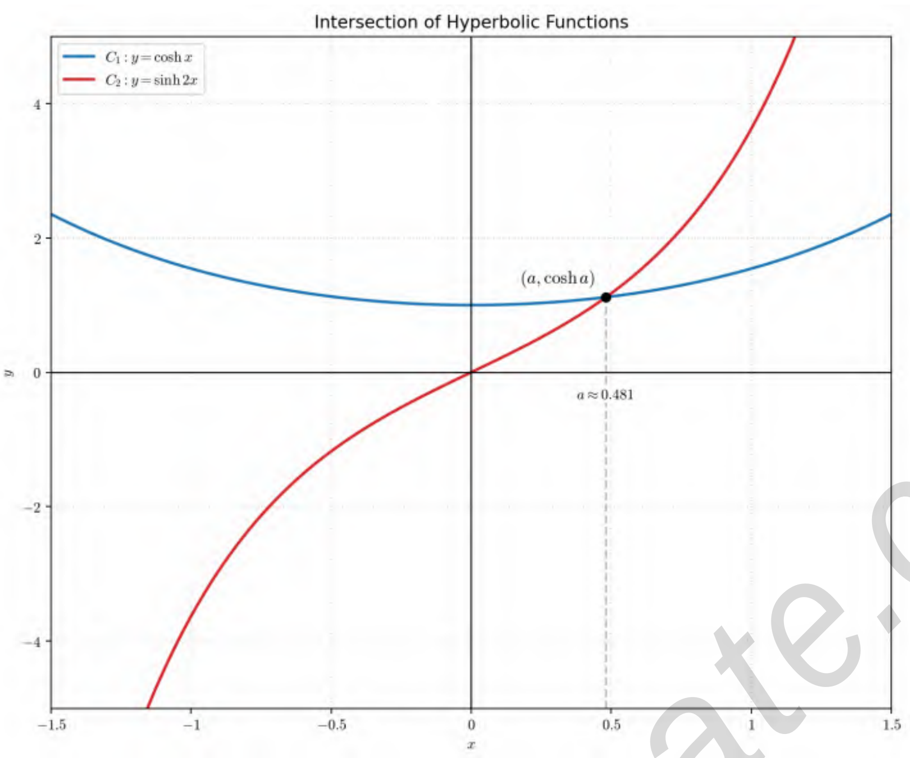
To find $x = a$ in logarithmic form, we use the inverse hyperbolic sine formula $\operatorname{arsinh} \ z = \ln(z + \sqrt{z^2 + 1})$:

$$\begin{aligned} a &= \ln\left(\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + 1}\right) \\ &= \ln\left(\frac{1}{2} + \sqrt{\frac{5}{4}}\right) \\ &= \ln\left(\frac{1 + \sqrt{5}}{2}\right) \end{aligned}$$

Note that this value is the logarithm of the **golden ratio** ϕ.

2. Sketching the curves

The curve $C_1 = \cosh x$ is an even function with a minimum at $(0, 1)$. The curve $C_2 = \sinh 2x$ is an odd function passing through the origin $(0, 0)$ and increasing rapidly for $x > 0$.



3. Calculating the arc length of C_1

The **arc length** s of a curve $y = f(x)$ from $x = x_1$ to $x = x_2$ is given by:

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For $C_1 : y = \cosh x$, the derivative is $\frac{dy}{dx} = \sinh x$. Substituting this into the formula:

$$\begin{aligned} s &= \int_0^a \sqrt{1 + \sinh^2 x}\, dx \\ &= \int_0^a \cosh x\, dx \end{aligned}$$

where we used the fundamental identity $\cosh^2 x - \sinh^2 x = 1$. Evaluating the integral:

$$\begin{aligned} s &= [\sinh x]_0^a \\ &= \sinh a - \sinh 0 \\ &= \sinh a \end{aligned}$$

From our calculation in step 1, we know that at the intersection point $x = a$, $\sinh a = \frac{1}{2}$.

(a) $\boxed{a = \ln\left(\frac{1 + \sqrt{5}}{2}\right)}$

(b) $\boxed{\frac{1}{2}}$

9231_21_Summer_2020_Q6

Solution

The integral is defined as $I_n = \int_0^{1/2} (1-x^2)^{-n/2} dx$.

1. Evaluation of I_1 To find I_1, we substitute $n = 1$ into the integral:

$$I_1 = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx$$

Using the standard integral for the **arcsine** function:

$$\begin{aligned} I_1 &= [\arcsin(x)]_0^{1/2} \\ &= \arcsin\left(\frac{1}{2}\right) - \arcsin(0) \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6} \end{aligned}$$

$\boxed{I_1 = \frac{\pi}{6}}$

2. Derivation of the Reduction Formula We consider the derivative of $f(x) = x(1-x^2)^{-n/2}$ using the **product rule** and **chain rule**:

$$\begin{aligned} \frac{d}{dx}\left(x(1-x^2)^{-n/2}\right) &= 1 \cdot (1-x^2)^{-n/2} + x \cdot \left(-\frac{n}{2}\right)(1-x^2)^{-n/2-1} \cdot (-2x) \\ &= (1-x^2)^{-n/2} + nx^2(1-x^2)^{-(n+2)/2} \end{aligned}$$

To relate this to I_n and I_{n+2}, we rewrite x^2 as $-(1-x^2)+1$:

$$\begin{aligned} \frac{d}{dx}\left(x(1-x^2)^{-n/2}\right) &= (1-x^2)^{-n/2} + n[-(1-x^2)+1](1-x^2)^{-(n+2)/2} \\ &= (1-x^2)^{-n/2} - n(1-x^2)^{-n/2} + n(1-x^2)^{-(n+2)/2} \\ &= (1-n)(1-x^2)^{-n/2} + n(1-x^2)^{-(n+2)/2} \end{aligned}$$

Integrating both sides from 0 to $1/2$:

$$\begin{aligned} \left[x(1-x^2)^{-n/2}\right]_0^{1/2} &= (1-n)I_n + nI_{n+2} \\ \frac{1}{2}\left(1-\frac{1}{4}\right)^{-n/2} - 0 &= (1-n)I_n + nI_{n+2} \\ \frac{1}{2}\left(\frac{3}{4}\right)^{-n/2} &= (1-n)I_n + nI_{n+2} \\ \frac{1}{2} \cdot 2^n \cdot 3^{-n/2} &= (1-n)I_n + nI_{n+2} \\ 2^{n-1}3^{-n/2} &= (1-n)I_n + nI_{n+2} \end{aligned}$$

Rearranging to the required form:

$$nI_{n+2} = 2^{n-1}3^{-n/2} + (n-1)I_n$$

3. Evaluation of I_5 We first need I_3. Setting $n = 1$ in the **reduction formula**:

$$\begin{aligned} 1 \cdot I_3 &= 2^{1-1}3^{-1/2} + (1-1)I_1 \\ I_3 &= 1 \cdot \frac{1}{\sqrt{3}} + 0 \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

Now, setting $n = 3$ to find I_5:

$$\begin{aligned} 3I_5 &= 2^{3-1}3^{-3/2} + (3-1)I_3 \\ 3I_5 &= 4 \cdot \frac{1}{3\sqrt{3}} + 2 \cdot \frac{\sqrt{3}}{3} \\ 3I_5 &= \frac{4\sqrt{3}}{9} + \frac{2\sqrt{3}}{3} \\ 3I_5 &= \frac{4\sqrt{3} + 6\sqrt{3}}{9} \\ 3I_5 &= \frac{10\sqrt{3}}{9} \\ I_5 &= \frac{10\sqrt{3}}{27} \end{aligned}$$

The value is in the form $k\sqrt{3}$ where $k = \frac{10}{27}$.

$$\boxed{I_5 = \frac{10}{27}\sqrt{3}}$$

9231_21_Summer_2020_Q7

Solution

1. Transformation of the Differential Equation

Given the substitution $x = t^3 y$, we determine the derivatives of x with respect to t using the **product rule**.

- First derivative:

$$\frac{dx}{dt} = t^3\frac{dy}{dt} + 3t^2 y$$

- Second derivative:

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(t^3\frac{dy}{dt} + 3t^2 y\right)\\ &= \left(t^3\frac{d^2y}{dt^2} + 3t^2\frac{dy}{dt}\right) + \left(3t^2\frac{dy}{dt} + 6ty\right)\\ &= t^3\frac{d^2y}{dt^2} + 6t^2\frac{dy}{dt} + 6ty\end{aligned}$$

Now, we substitute these expressions into the linear combination $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x$:

$$\begin{aligned}\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x &= \left(t^3\frac{d^2y}{dt^2} + 6t^2\frac{dy}{dt} + 6ty\right) + 4\left(t^3\frac{dy}{dt} + 3t^2 y\right) + 13(t^3 y)\\ &= t^3\frac{d^2y}{dt^2} + (4t^3 + 6t^2)\frac{dy}{dt} + (13t^3 + 12t^2 + 6t)y\end{aligned}$$

Comparing this result to the original differential equation provided:

$$t^3\frac{d^2y}{dt^2} + (4t^3 + 6t^2)\frac{dy}{dt} + (13t^3 + 12t^2 + 6t)y = 61e^{\frac{1}{2}t}$$

It is shown that:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 61e^{\frac{1}{2}t}$$

2. Solving for the General Solution

To find $y(t)$, we first solve the second-order linear **non-homogeneous differential equation** for $x(t)$.

- **Complementary Function (x_c):** The **characteristic equation** is $r^2 + 4r + 13 = 0$.

$$\begin{aligned} r &= \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2} \\ &= \frac{-4 \pm \sqrt{16 - 52}}{2} \\ &= \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i \end{aligned}$$

Thus, $x_c(t) = e^{-2t}(A\cos(3t) + B\sin(3t))$.

- **Particular Integral (x_p):** Assume a solution of the form $x_p(t) = Ce^{\frac{1}{2}t}$.

$$\frac{dx_p}{dt} = \frac{1}{2}Ce^{\frac{1}{2}t}$$
$$\frac{d^2x_p}{dt^2} = \frac{1}{4}Ce^{\frac{1}{2}t}$$

Substituting into the differential equation for x:

$$\begin{aligned} \left(\frac{1}{4}C + 4\left(\frac{1}{2}C\right) + 13C\right)e^{\frac{1}{2}t} &= 61e^{\frac{1}{2}t} \\ \left(\frac{1}{4} + 2 + 13\right)C &= 61 \\ \frac{61}{4}C &= 61 \implies C = 4 \end{aligned}$$

So, $x_p(t) = 4e^{\frac{1}{2}t}$.

- **General Solution for $x(t)$:**

$$x(t) = e^{-2t}(A\cos(3t) + B\sin(3t)) + 4e^{\frac{1}{2}t}$$

- **General Solution for $y(t)$:** Using the relation $y = \frac{x}{t^3}$:

$$y = \frac{e^{-2t}(A\cos(3t) + B\sin(3t)) + 4e^{\frac{1}{2}t}}{t^3}$$

$$\boxed{y = \frac{1}{t^3}\left[e^{-2t}(A\cos(3t) + B\sin(3t)) + 4e^{\frac{1}{2}t}\right]}$$

9231_21_Summer_2020_Q8

Solution

1. Non-unique solution for the system of equations

A **homogeneous system** of linear equations has a non-unique solution (i.e., non-trivial solutions exist) if and only if the **determinant** of its coefficient matrix is zero. The system is:

$$\begin{gathered}\{3x + y + z = 0 \\ ax + 6y - z = 0 \\ 0x + ay - 2z = 0\end{gathered}$$

The coefficient matrix M is:

$$M = (3 \;\; 1 \;\; 1; a \;\; 6 \;\; -1; 0 \;\; a \;\; -2)$$

We calculate the determinant $\det(M)$ using **Laplace expansion** along the first column:

$$\begin{aligned}\det(M) &= 3|6 \;\; -1; a \;\; -2| - a|1 \;\; 1; a \;\; -2| + 0|1 \;\; 1; 6 \;\; -1| \\ &= 3(-12 - (-a)) - a(-2 - a) \\ &= 3(a - 12) + a(a + 2) \\ &= 3a - 36 + a^2 + 2a \\ &= a^2 + 5a - 36\end{aligned}$$

For a non-unique solution, we set $\det(M) = 0$:

$$\begin{gathered}a^2 + 5a - 36 = 0 \\ (a + 9)(a - 4) = 0\end{gathered}$$

Thus, the values of a are $a = -9$ and $a = 4$.

$\boxed{a = -9, 4}$

2. Inverse of A^2 using the characteristic equation

The matrix A is an **upper triangular matrix**:

$$A = (3 \;\; 1 \;\; 1; 0 \;\; 6 \;\; -1; 0 \;\; 0 \;\; -2)$$

The **eigenvalues** are the diagonal entries: $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = -2$. The **characteristic equation** is given by $p(\lambda) = \det(A - \lambda I) = 0$:

$$\begin{aligned}(3 - \lambda)(6 - \lambda)(-2 - \lambda) &= 0 \\ (\lambda^2 - 9\lambda + 18)(-2 - \lambda) &= 0 \\ -2\lambda^2 - \lambda^3 + 18\lambda + 9\lambda^2 - 36 - 18\lambda &= 0 \\ -\lambda^3 + 7\lambda^2 - 36 &= 0 \\ \lambda^3 - 7\lambda^2 + 36 &= 0\end{aligned}$$

By the **Cayley-Hamilton Theorem**, A satisfies its own characteristic equation:

$$A^3 - 7A^2 + 36I = 0$$

To find $(A^2)^{-1}$, we first find A^{-1}. Multiplying by A^{-1}:

$$\begin{aligned} A^2 - 7A + 36A^{-1} &= 0 \\ 36A^{-1} &= 7A - A^2 \\ A^{-1} &= \frac{1}{36}(7A - A^2) \end{aligned}$$

Then $(A^2)^{-1} = (A^{-1})^2$. Alternatively, from $A^3 - 7A^2 + 36I = 0$, we can write:

$$A^2(A - 7I) = -36I$$

$$A^2\left(\frac{7I - A}{36}\right) = I$$

Thus, $(A^2)^{-1} = \frac{1}{36}(7I - A)$. Calculating $7I - A$:

$$7I - A = (7\ \ 0\ \ 0;0\ \ 7\ \ 0;0\ \ 0\ \ 7) - (3\ \ 1\ \ 1;0\ \ 6\ \ -1;0\ \ 0\ \ -2) = (4\ \ -1\ \ -1;0\ \ 1\ \ 1;0\ \ 0\ \ 9)$$

$$(A^2)^{-1} = \frac{1}{36}(4\ \ -1\ \ -1;0\ \ 1\ \ 1;0\ \ 0\ \ 9)$$

$$\boxed{(A^2)^{-1} = \begin{pmatrix} 1/9 & -1/36 & -1/36 \\ 0 & 1/36 & 1/36 \\ 0 & 0 & 1/4 \end{pmatrix}}$$

3. Diagonalization and A^5

To find P and D such that $A^5 = PDP^{-1}$, we use **diagonalization**. Since A has distinct eigenvalues, it is diagonalizable. The matrix D will contain the eigenvalues of A^5 on its diagonal:

$$D = (3^5\ \ 0\ \ 0;0\ \ 6^5\ \ 0;0\ \ 0\ \ (-2)^5) = (243\ \ 0\ \ 0;0\ \ 7776\ \ 0;0\ \ 0\ \ -32)$$

The matrix P consists of the **eigenvectors** of A.

- For $\lambda_1 = 3$: $(A - 3I)\mathbf{v}_1 = 0 \implies (0\ \ 1\ \ 1;0\ \ 3\ \ -1;0\ \ 0\ \ -5)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \implies y = 0, z = 0$. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
- For $\lambda_2 = 6$: $(A - 6I)\mathbf{v}_2 = 0 \implies (-3\ \ 1\ \ 1;0\ \ 0\ \ -1;0\ \ 0\ \ -8)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \implies z = 0, -3x + y = 0$. Let $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.
- For $\lambda_3 = -2$: $(A + 2I)\mathbf{v}_3 = 0 \implies (5\ \ 1\ \ 1;0\ \ 8\ \ -1;0\ \ 0\ \ 0)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$. From row 2: $8y = z$. From row 1: $5x + y + 8y = 0 \implies 5x = -9y$. Let $y = 5$, then $z = 40, x = -9$. Let $\mathbf{v}_3 = \begin{pmatrix} -9 \\ 5 \\ 40 \end{pmatrix}$.

Thus, $P = (1\ \ 1\ \ -9;0\ \ 3\ \ 5;0\ \ 0\ \ 40)$.

$$\boxed{P = \begin{pmatrix} 1 & 1 & -9 \\ 0 & 3 & 5 \\ 0 & 0 & 40 \end{pmatrix}, D = \begin{pmatrix} 243 & 0 & 0 \\ 0 & 7776 & 0 \\ 0 & 0 & -32 \end{pmatrix}}$$