

9231_11_Summer_2020_Q1

Solution

1. Analysis and Sketch of the Rational Function

To sketch the curve $y = \frac{ax}{x+7}$ where $a > 0$, we identify the key features of this **rational function**.

- **Asymptotes:**

- ▶ The **vertical asymptote** occurs where the denominator is zero: $x + 7 = 0 \implies x = -7$.
- ▶ The **horizontal asymptote** is found by taking the limit as $x \rightarrow \pm\infty$:

$$\lim_{x \rightarrow \pm\infty} \frac{ax}{x+7} = \lim_{x \rightarrow \pm\infty} \frac{a}{1 + \frac{7}{x}} = a$$

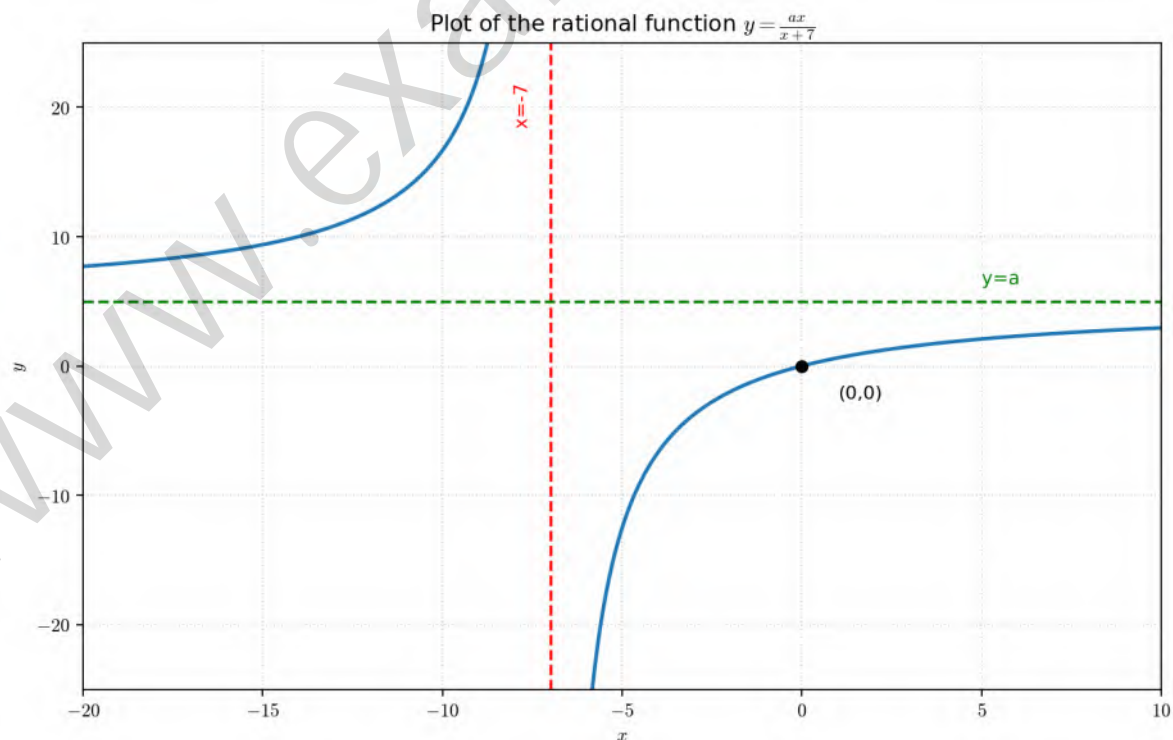
Thus, the horizontal asymptote is $y = a$.

- **Intercepts:**

- ▶ The x -intercept occurs when $y = 0$: $ax = 0 \implies x = 0$.
- ▶ The y -intercept occurs when $x = 0$: $y = \frac{a(0)}{0+7} = 0$.
- ▶ The curve passes through the origin $(0, 0)$.

- **Behavior:**

- ▶ For $x > -7$, as $x \rightarrow -7^+$, $y \rightarrow -\infty$.
- ▶ For $x < -7$, as $x \rightarrow -7^-$, $y \rightarrow +\infty$.
- ▶ The derivative $y' = \frac{a(x+7) - ax}{(x+7)^2} = \frac{7a}{(x+7)^2}$ is always positive for $x \neq -7$, meaning the function is strictly increasing on its domain.

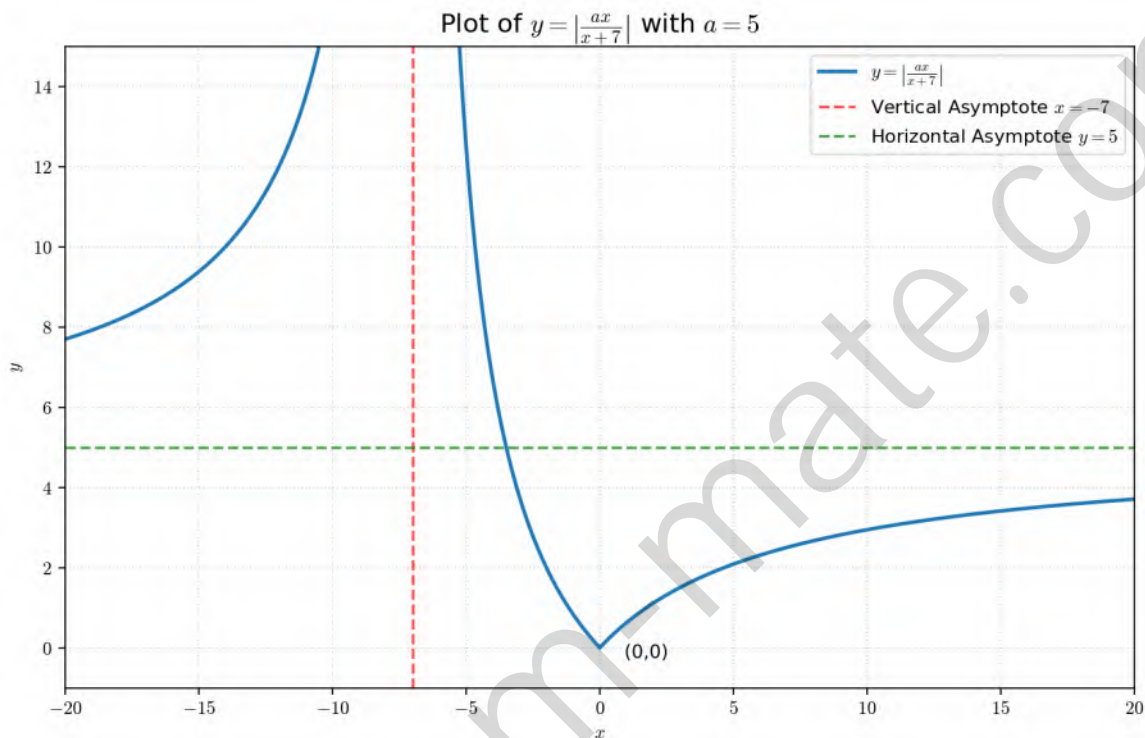


2. Analysis of the Absolute Value Function

The curve $y = \left| \frac{ax}{x+7} \right|$ is obtained by reflecting the portions of the graph where $y < 0$ (the part between $x = -7$ and $x = 0$) across the x -axis.

• **Features:**

- Vertical asymptote remains at $x = -7$.
- Horizontal asymptote remains at $y = a$.
- The function is non-negative ($y \geq 0$) for all x in the domain.
- The point $(0, 0)$ becomes a sharp corner (cusp).



3. Solving the Inequality

We seek the set of values for x such that $\left| \frac{ax}{x+7} \right| > \frac{a}{2}$. Since $a > 0$, we can divide both sides by a :

$$\left| \frac{x}{x+7} \right| > \frac{1}{2}$$

This **absolute value inequality** is equivalent to:

$$\frac{x}{x+7} > \frac{1}{2} \quad \text{or} \quad \frac{x}{x+7} < -\frac{1}{2}$$

• **Case 1:** $\frac{x}{x+7} > \frac{1}{2}$

$$\frac{x}{x+7} - \frac{1}{2} > 0$$

$$\frac{2x - (x+7)}{2(x+7)} > 0$$

$$\frac{x-7}{2(x+7)} > 0$$

The critical points are $x = 7$ and $x = -7$. Testing intervals:

- $x < -7$: Positive.
- $-7 < x < 7$: Negative.
- $x > 7$: Positive. So, $x < -7$ or $x > 7$.

• **Case 2:** $\frac{x}{x+7} < -\frac{1}{2}$

$$\begin{aligned}\frac{x}{x+7} + \frac{1}{2} &< 0 \\ \frac{2x + (x+7)}{2(x+7)} &< 0 \\ \frac{3x+7}{2(x+7)} &< 0\end{aligned}$$

The critical points are $x = -7$ and $x = -\frac{7}{3}$. Testing intervals:

- $x < -7$: Positive.
- $-7 < x < -\frac{7}{3}$: Negative.
- $x > -\frac{7}{3}$: Positive. So, $-7 < x < -\frac{7}{3}$.

Combining the results from both cases, the set of values for x is:

$$\left\{ x \in \mathbb{R} : x < -7 \text{ or } -7 < x < -\frac{7}{3} \text{ or } x > 7 \right\}$$

$x < -7 \cup -7 < x < -\frac{7}{3} \cup x > 7$

9231_11_Summer_2020_Q2

Solution

1. Finding the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$

To find a cubic equation whose roots are the squares of the roots of the original equation $6x^3 + px^2 - 3x - 5 = 0$, we use the method of **transformation of equations**. Let $y = x^2$, which implies $x = \pm\sqrt{y}$. Substituting this into the original equation:

$$6(\pm\sqrt{y})^3 + p(\pm\sqrt{y})^2 - 3(\pm\sqrt{y}) - 5 = 0$$

$$\pm 6y\sqrt{y} + py \mp 3\sqrt{y} - 5 = 0$$

Rearranging to group the terms with \sqrt{y} on one side:

$$\pm\sqrt{y}(6y - 3) = 5 - py$$

Squaring both sides to eliminate the radical:

$$y(6y - 3)^2 = (5 - py)^2$$

$$y(36y^2 - 36y + 9) = 25 - 10py + p^2y^2$$

$$36y^3 - 36y^2 + 9y = 25 - 10py + p^2y^2$$

Rearranging into standard cubic form $Ay^3 + By^2 + Cy + D = 0$:

$$36y^3 - (36 + p^2)y^2 + (9 + 10p)y - 25 = 0$$

Thus, the cubic equation in x with roots $\alpha^2, \beta^2, \gamma^2$ is:

$$\boxed{36x^3 - (36 + p^2)x^2 + (9 + 10p)x - 25 = 0}$$

2. Finding the value of p

From **Vieta's formulas** applied to the original equation $6x^3 + px^2 - 3x - 5 = 0$:

- $\alpha + \beta + \gamma = -\frac{p}{6}$

From the transformed equation $36x^3 - (36 + p^2)x^2 + (9 + 10p)x - 25 = 0$, the sum of the roots is:

- $\alpha^2 + \beta^2 + \gamma^2 = \frac{36 + p^2}{36}$

We are given the condition $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$. Substituting the expressions above:

$$\frac{36 + p^2}{36} = 2\left(-\frac{p}{6}\right)$$

$$\frac{36 + p^2}{36} = -\frac{p}{3}$$

Multiplying both sides by 36:

$$36 + p^2 = -12p$$

$$p^2 + 12p + 36 = 0$$

$$(p + 6)^2 = 0$$

$$\boxed{p = -6}$$

3. Finding the value of $\alpha^3 + \beta^3 + \gamma^3$

Since α, β, γ are roots of $6x^3 - 6x^2 - 3x - 5 = 0$ (substituting $p = -6$), each root satisfies:

$$6x^3 = 6x^2 + 3x + 5$$

$$x^3 = x^2 + \frac{1}{2}x + \frac{5}{6}$$

Summing this relation for all three roots:

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha^2 + \beta^2 + \gamma^2) + \frac{1}{2}(\alpha + \beta + \gamma) + 3\left(\frac{5}{6}\right)$$

Using the values found previously with $p = -6$:

- $\alpha + \beta + \gamma = -\frac{-6}{6} = 1$
- $\alpha^2 + \beta^2 + \gamma^2 = 2(1) = 2$

Substituting these into the sum:

$$\begin{aligned}\alpha^3 + \beta^3 + \gamma^3 &= 2 + \frac{1}{2}(1) + \frac{15}{6} \\ &= 2 + 0.5 + 2.5 \\ &= 5\end{aligned}$$

$$\boxed{\alpha^3 + \beta^3 + \gamma^3 = 5}$$

9231_11_Summer_2020_Q3

Solution

The curve C is defined by the **rational function**:

$$y = \frac{x^2}{2x + 1}$$

1. Finding the Equations of the Asymptotes

- **Vertical Asymptote** A **vertical asymptote** occurs where the denominator is zero and the numerator is non-zero.

$$\begin{aligned} 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Since the numerator $x^2 = (-1/2)^2 = 1/4 \neq 0$ at this point, $x = -1/2$ is a vertical asymptote.

- **Oblique Asymptote** Since the degree of the numerator is exactly one higher than the degree of the denominator, there is an **oblique asymptote** (or slant asymptote). We find it using **polynomial long division**:

$$\begin{aligned} \frac{x^2}{2x + 1} &= \frac{\frac{1}{2}x(2x + 1) - \frac{1}{2}x}{2x + 1} \\ &= \frac{1}{2}x - \frac{\frac{1}{2}x}{2x + 1} \\ &= \frac{1}{2}x - \frac{\frac{1}{4}(2x + 1) - \frac{1}{4}}{2x + 1} \\ &= \frac{1}{2}x - \frac{1}{4} + \frac{1}{4(2x + 1)} \end{aligned}$$

As $x \rightarrow \pm\infty$, the term $\frac{1}{4(2x+1)} \rightarrow 0$. Thus, the oblique asymptote is $y = \frac{1}{2}x - \frac{1}{4}$.

2. Finding the Coordinates of the Stationary Points

To find the **stationary points**, we set the first derivative dy/dx to zero using the **quotient rule**:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x + 1)\frac{d}{dx}(x^2) - (x^2)\frac{d}{dx}(2x + 1)}{(2x + 1)^2} \\ &= \frac{(2x + 1)(2x) - (x^2)(2)}{(2x + 1)^2} \\ &= \frac{4x^2 + 2x - 2x^2}{(2x + 1)^2} \\ &= \frac{2x^2 + 2x}{(2x + 1)^2} \end{aligned}$$

Setting $dy/dx = 0$:

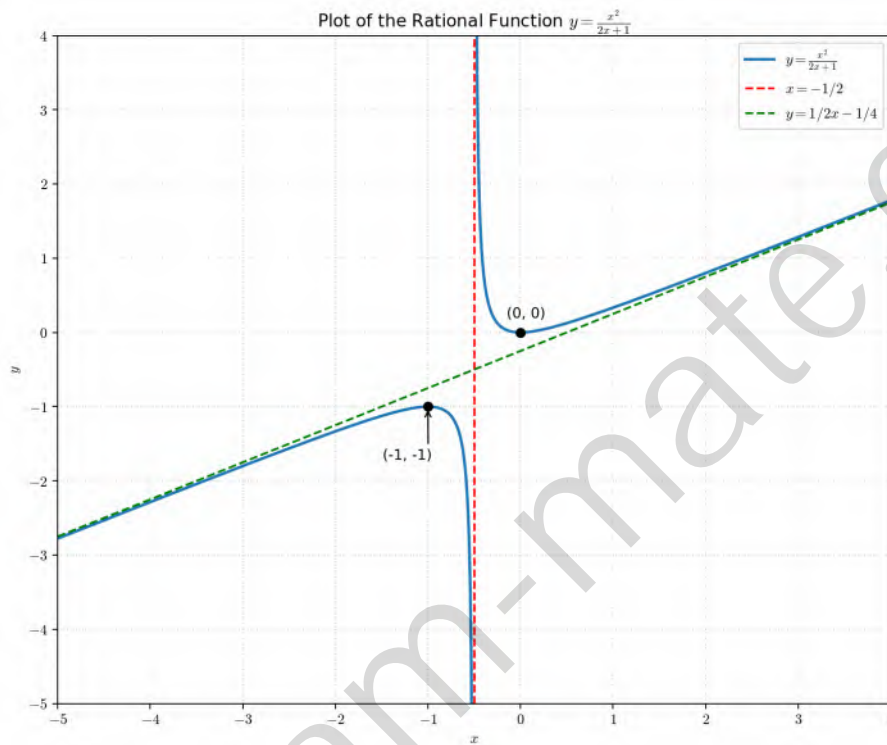
$$2x^2 + 2x = 0$$

$$2x(x + 1) = 0$$

This gives $x = 0$ and $x = -1$. Now, find the corresponding y -coordinates:

- For $x = 0$: $y = \frac{0^2}{2(0)+1} = 0$. Point: $(0, 0)$.
- For $x = -1$: $y = \frac{(-1)^2}{2(-1)+1} = \frac{1}{-1} = -1$. Point: $(-1, -1)$.

3. Sketching the Curve



- The curve passes through the origin $(0, 0)$, which is a local minimum.
- There is a local maximum at $(-1, -1)$.
- The curve is confined by the vertical asymptote $x = -0.5$ and approaches the line $y = 0.5x - 0.25$ for large $|x|$.

Final Answers:

(a) Asymptotes: $x = -\frac{1}{2}, y = \frac{1}{2}x - \frac{1}{4}$

(b) Stationary points: $(0, 0)$ and $(-1, -1)$

9231_11_Summer_2020_Q4

Solution

1. Partial Fraction Decomposition

To express the general term in **partial fractions**, we factor the denominator:

$$r^2 - 1 = (r - 1)(r + 1)$$

We set up the identity:

$$\frac{1}{(r - 1)(r + 1)} = \frac{A}{r - 1} + \frac{B}{r + 1}$$

Multiplying by the common denominator gives $1 = A(r + 1) + B(r - 1)$.

- Setting $r = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$
- Setting $r = -1$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

Thus, the partial fraction form is:

$$\frac{1}{r^2 - 1} = \frac{1}{2} \left(\frac{1}{r - 1} - \frac{1}{r + 1} \right)$$

2. Summation using the Method of Differences

We evaluate the finite sum $S_n = \sum_{r=2}^n \frac{1}{r^2-1}$ using the **method of differences** (telescoping sum):

$$\begin{aligned} S_n &= \frac{1}{2} \sum_{r=2}^n \left(\frac{1}{r-1} - \frac{1}{r+1} \right) \\ &= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \left(\frac{1}{n-2} - \frac{1}{n} \right) + \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right] \end{aligned}$$

Most terms cancel out, leaving:

$$\begin{aligned} S_n &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{2} \left(\frac{3}{2} - \frac{(n+1) + n}{n(n+1)} \right) \\ &= \frac{3}{4} - \frac{2n+1}{2n(n+1)} \end{aligned}$$

Comparing this to the required form $\frac{3}{4} - \frac{an+b}{2n(n+1)}$, we find: $a = 2, b = 1$

3. Infinite Series Sum

To find the value of the **infinite series** $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$, we take the limit of the partial sum S_n as $n \rightarrow \infty$:

$$\begin{aligned}\sum_{r=2}^{\infty} \frac{1}{r^2 - 1} &= \lim_{n \rightarrow \infty} \left(\frac{3}{4} - \frac{2n+1}{2n^2+2n} \right) \\ &= \frac{3}{4} - 0\end{aligned}$$

$\frac{3}{4}$

4. Limit of the Tail Sum

We are asked to find the limit as $n \rightarrow \infty$ of:

$$T_n = \sum_{r=n+1}^{2n} \frac{n}{r^2 - 1}$$

We can rewrite this sum using the result from part (a). Note that $\sum_{r=n+1}^{2n} f(r) = S_{2n} - S_n$:

$$\begin{aligned}T_n &= n \left(\sum_{r=2}^{2n} \frac{1}{r^2 - 1} - \sum_{r=2}^n \frac{1}{r^2 - 1} \right) \\ &= n \left[\left(\frac{3}{4} - \frac{2(2n)+1}{2(2n)(2n+1)} \right) - \left(\frac{3}{4} - \frac{2n+1}{2n(n+1)} \right) \right] \\ &= n \left[\frac{2n+1}{2n(n+1)} - \frac{4n+1}{4n(2n+1)} \right] \\ &= \frac{2n+1}{2(n+1)} - \frac{4n+1}{4(2n+1)}\end{aligned}$$

Now, we take the **limit** as $n \rightarrow \infty$:

$$\begin{aligned}\lim_{n \rightarrow \infty} T_n &= \lim_{n \rightarrow \infty} \frac{2n+1}{2n+2} - \lim_{n \rightarrow \infty} \frac{4n+1}{8n+4} \\ &= \frac{2}{2} - \frac{4}{8} \\ &= 1 - \frac{1}{2}\end{aligned}$$

$\frac{1}{2}$

9231_11_Summer_2020_Q5

Solution

The problem involves finding the shortest distance between two skew lines, the equation of a plane containing a line and parallel to a vector, and the angle between a line and a plane.

1. Shortest distance between l_1 and l_2

The equations of the lines are:

$$l_1 : \mathbf{r}_1 = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

$$l_2 : \mathbf{r}_2 = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Let $\mathbf{a}_1 = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 3 \\ -5 \\ -6 \end{pmatrix}$, and $\mathbf{d}_2 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$. The **shortest distance** D between two skew lines is given by the projection of the vector connecting points on the lines onto the common perpendicular:

$$D = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$$

- Calculate the **cross product** $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 4 \\ 0 & 5 & 6 \end{vmatrix} = \mathbf{i}(24 - 20) - \mathbf{j}(6 - 0) + \mathbf{k}(5 - 0) = \begin{pmatrix} 4 \\ -6 \\ 5 \end{pmatrix}$$

- Calculate the magnitude $|\mathbf{n}|$:

$$|\mathbf{n}| = \sqrt{4^2 + (-6)^2 + 5^2} = \sqrt{16 + 36 + 25} = \sqrt{77}$$

- Calculate the vector $\mathbf{a}_2 - \mathbf{a}_1$:

$$\mathbf{a}_2 - \mathbf{a}_1 = \begin{pmatrix} 3 - 3 \\ -5 - 0 \\ -6 - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -9 \end{pmatrix}$$

- Calculate the **dot product** $(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{n}$:

$$(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{n} = (0)(4) + (-5)(-6) + (-9)(5) = 0 + 30 - 45 = -15$$

- The shortest distance is:

$$D = \frac{|-15|}{\sqrt{77}} = \frac{15}{\sqrt{77}} \approx 1.71$$

2. Equation of the plane Π

The plane Π contains l_1 , so it contains the point $(3, 0, 3)$ and is parallel to the direction vector $\mathbf{d}_1 = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$. It is also parallel to the vector $\mathbf{v} = \mathbf{i} + \mathbf{k} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The **normal vector** \mathbf{n}_Π to the plane is:

$$\mathbf{n}_\Pi = \mathbf{d}_1 \times \mathbf{v} = |\mathbf{i} \ \mathbf{j} \ \mathbf{k}; 1 \ 4 \ 4; 1 \ 0 \ 1| = \mathbf{i}(4-0) - \mathbf{j}(1-4) + \mathbf{k}(0-4) = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$$

The equation of the plane in the form $ax + by + cz = d$ is:

$$4x + 3y - 4z = d$$

Substitute the point $(3, 0, 3)$ to find d :

$$4(3) + 3(0) - 4(3) = 12 - 12 = 0$$

Thus, the equation of Π is $4x + 3y - 4z = 0$.

3. Acute angle between l_2 and Π

The **angle** θ between a line with direction \mathbf{d} and a plane with normal \mathbf{n} is given by:

$$\sin \theta = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}| |\mathbf{n}|}$$

For l_2 , $\mathbf{d}_2 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$. For Π , $\mathbf{n}_\Pi = \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$.

- Calculate the dot product:

$$\mathbf{d}_2 \cdot \mathbf{n}_\Pi = (0)(4) + (5)(3) + (6)(-4) = 0 + 15 - 24 = -9$$

- Calculate the magnitudes:

$$|\mathbf{d}_2| = \sqrt{0^2 + 5^2 + 6^2} = \sqrt{25 + 36} = \sqrt{61}$$

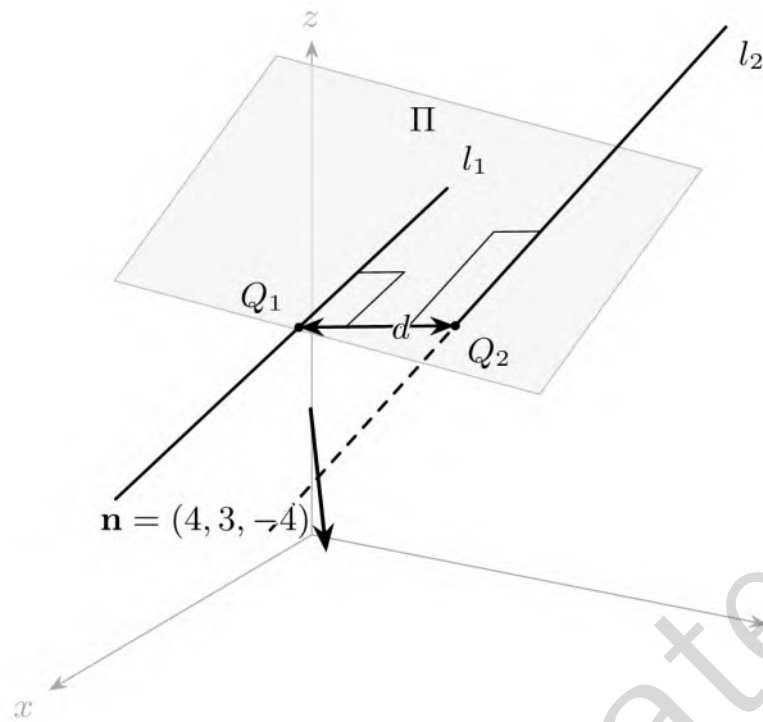
$$|\mathbf{n}_\Pi| = \sqrt{4^2 + 3^2 + (-4)^2} = \sqrt{16 + 9 + 16} = \sqrt{41}$$

- Calculate $\sin \theta$:

$$\sin \theta = \frac{|-9|}{\sqrt{61}\sqrt{41}} = \frac{9}{\sqrt{2501}}$$

- Find the angle θ :

$$\theta = \arcsin\left(\frac{9}{\sqrt{2501}}\right) \approx 10.36^\circ$$



- (a) Shortest distance: $\frac{15}{\sqrt{77}}$
 (b) Equation of Π : $4x + 3y - 4z = 0$
 (c) Acute angle: 10.4° (to 1 d.p.)

Final Answers: (a) $\frac{15}{\sqrt{77}}$ (b) $4x + 3y - 4z = 0$ (c) 10.4°

9231_11_Summer_2020_Q6

Solution

1. Determinant and Area Transformation

The area of a shape transformed by a matrix M is given by the product of the original area and the absolute value of the **determinant** of M . Given the matrix $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$, we first find its determinant:

$$\det(A) = (2)(1) - (0)(1) = 2$$

The transformation is represented by A^{-1} . Using the property $\det(A^{-1}) = \frac{1}{\det(A)}$, we have:

$$\det(A^{-1}) = \frac{1}{2}$$

The new area d is calculated as:

$$\begin{aligned} d &= \text{Area}_{\text{original}} \times |\det(A^{-1})| \\ &= 30 \text{ cm}^2 \times \frac{1}{2} \\ &= 15 \text{ cm}^2 \end{aligned}$$

The value of d is 15.

2. Proof by Mathematical Induction

We wish to prove that $A^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$ for all positive integers n .

- **Base Case:** For $n = 1$:

$$A^1 = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

This matches the given matrix A , so the base case holds.

- **Inductive Step:** Assume the statement is true for $n = k$, i.e., $A^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$. We must show it holds for $n = k + 1$:

$$\begin{aligned} A^{k+1} &= A^k \cdot A \\ &= \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \left((2^k)(2) + (0)(1) \quad (2^k)(0) + (0)(1); (2^k - 1)(2) + (1)(1) \quad (2^k - 1)(0) + (1)(1) \right) \\ &= \begin{pmatrix} 2^{k+1} & 0 \\ 2 \cdot 2^k - 2 + 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix} \end{aligned}$$

The formula holds for $n = k + 1$. By the principle of **mathematical induction**, the statement is true for all positive integers n .

3. Invariant Line under Matrix Transformation

Let $M = A^n B$. We first compute the product M :

$$\begin{aligned}
 M &= \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} (2^n)(1) + (0)(33) & (2^n)(0) + (0)(0) \\ (2^n - 1)(1) + (1)(33) & (2^n - 1)(0) + (1)(0) \end{pmatrix} \\
 &= \begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}
 \end{aligned}$$

A line $y = mx$ is an **invariant line** if any point (x, y) on the line is mapped to a point (x', y') that also lies on the line. For the line $y = 2x$, we transform a general point $(x, 2x)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix} \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 2^n x \\ (2^n + 32)x \end{pmatrix}$$

For the line to be invariant, the condition $y' = 2x'$ must be satisfied:

$$\begin{aligned}
 (2^n + 32)x &= 2(2^n x) \\
 2^n + 32 &= 2 \cdot 2^n \\
 32 &= 2 \cdot 2^n - 2^n \\
 32 &= 2^n \\
 2^5 &= 2^n
 \end{aligned}$$

Thus, $n = 5$.

The value of n is .

9231_11_Summer_2020_Q7

Solution

1. Analysis of Curve C_1 and the Point P

The curve C_1 is defined by the **polar equation** $r = \theta \cos \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. The distance of a point (r, θ) from the line $\theta = \frac{\pi}{2}$ (the y -axis) is given by the absolute value of its x -coordinate. In **polar coordinates**, $x = r \cos \theta$.

- **Deriving the condition for P :** Substituting the expression for r :

$$x = (\theta \cos \theta) \cos \theta = \theta \cos^2 \theta$$

To find the point P furthest from the y -axis, we maximize x with respect to θ by setting the derivative $\frac{dx}{d\theta} = 0$:

$$\begin{aligned} \frac{dx}{d\theta} &= \frac{d}{d\theta}(\theta \cos^2 \theta) \\ &= 1 \cdot \cos^2 \theta + \theta \cdot 2 \cos \theta (-\sin \theta) \\ &= \cos^2 \theta - 2\theta \sin \theta \cos \theta \end{aligned}$$

Setting $\frac{dx}{d\theta} = 0$:

$$\begin{aligned} \cos^2 \theta - 2\theta \sin \theta \cos \theta &= 0 \\ \cos \theta (\cos \theta - 2\theta \sin \theta) &= 0 \end{aligned}$$

Since $0 \leq \theta \leq \frac{\pi}{2}$, $\cos \theta = 0$ only at $\theta = \frac{\pi}{2}$ (where $x = 0$, a minimum). For the maximum, we solve:

$$\begin{aligned} \cos \theta - 2\theta \sin \theta &= 0 \\ 1 - 2\theta \frac{\sin \theta}{\cos \theta} &= 0 \\ 1 - 2\theta \tan \theta &= 0 \\ 2\theta \tan \theta - 1 &= 0 \end{aligned}$$

- **Verification of the root:** Let $f(\theta) = 2\theta \tan \theta - 1$. We use the **Intermediate Value Theorem**:

- $f(0.6) = 2(0.6) \tan(0.6) - 1 \approx -0.1790$
- $f(0.7) = 2(0.7) \tan(0.7) - 1 \approx 0.1791$ Since $f(0.6) < 0$ and $f(0.7) > 0$, and f is continuous on $[0.6, 0.7]$, there exists a root in $(0.6, 0.7)$.

2. Intersection Point Q

The curves are $C_1 : r = \theta \cos \theta$ and $C_2 : r = \theta \sin \theta$. To find the intersection point Q (other than the pole O where $r = 0$):

$$\begin{aligned} \theta \cos \theta &= \theta \sin \theta \\ \cos \theta &= \sin \theta \\ \tan \theta &= 1 \end{aligned}$$

For $0 \leq \theta \leq \frac{\pi}{2}$, this gives $\theta = \frac{\pi}{4}$. Substituting $\theta = \frac{\pi}{4}$ into C_1 :

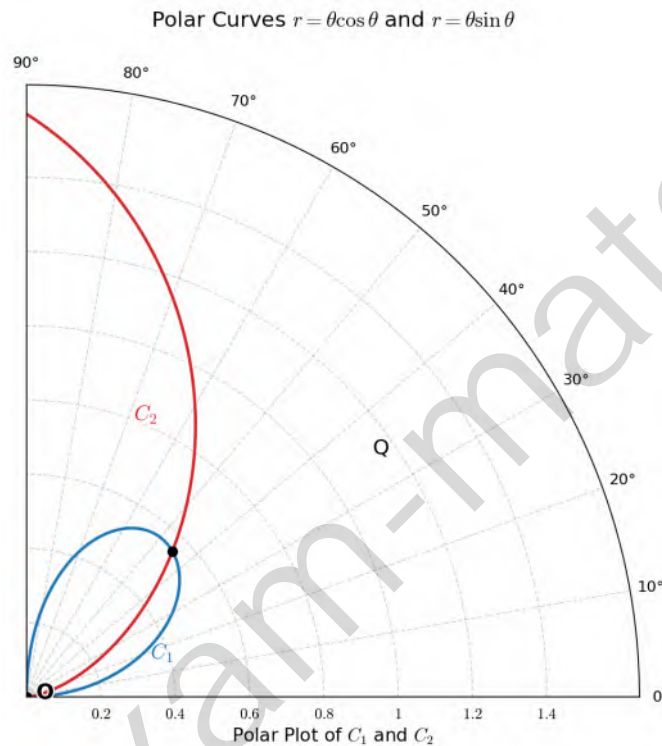
$$r = \frac{\pi}{4} \cos\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{8}$$

$$\left(\frac{\pi\sqrt{2}}{8}, \frac{\pi}{4}\right)$$

The coordinates of Q are

3. Sketch of C_1 and C_2

C_1 starts at the pole ($r = 0$) at $\theta = 0$, reaches a maximum x at P , and returns to the pole at $\theta = \frac{\pi}{2}$. C_2 starts at the pole at $\theta = 0$ and returns to the pole at $\theta = \frac{\pi}{2}$ (since $\sin(0) = 0$ and $\theta = 0$ at the start).



4. Area of the Bounded Region

The area A in polar coordinates is given by $\int \frac{1}{2} r^2 d\theta$. The region is bounded by C_2 from $\theta = 0$ to $\theta = \frac{\pi}{4}$ and by C_1 from $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$ (or vice versa depending on the orientation, but here C_1 is "above" C_2 for $0 < \theta < \frac{\pi}{4}$). However, the region bounded by the arcs OQ of both curves is the area between them. For $0 \leq \theta \leq \frac{\pi}{4}$, C_1 is the outer curve ($r_1 = \theta \cos \theta \geq \theta \sin \theta = r_2$).

$$\begin{aligned} \text{Area} &= \int_0^{\pi/4} \frac{1}{2} (r_1^2 - r_2^2) d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} (\theta^2 \cos^2 \theta - \theta^2 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \theta^2 (\cos^2 \theta - \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \theta^2 \cos(2\theta) d\theta \end{aligned}$$

Using **integration by parts** twice ($\int u dv = uv - \int v du$): Let $u = \theta^2$, $dv = \cos(2\theta) d\theta \implies du = 2\theta d\theta$, $v = \frac{1}{2} \sin(2\theta)$.

$$\int \theta^2 \cos(2\theta) d\theta = \frac{\theta^2}{2} \sin(2\theta) - \int \theta \sin(2\theta) d\theta$$

For $\int \theta \sin(2\theta) d\theta$, let $u = \theta$, $dv = \sin(2\theta) d\theta \implies du = d\theta$, $v = -\frac{1}{2} \cos(2\theta)$.

$$\begin{aligned} \int \theta \sin(2\theta) d\theta &= -\frac{\theta}{2} \cos(2\theta) + \int \frac{1}{2} \cos(2\theta) d\theta \\ &= -\frac{\theta}{2} \cos(2\theta) + \frac{1}{4} \sin(2\theta) \end{aligned}$$

Combining the results:

$$\begin{aligned} \text{Area} &= \frac{1}{2} \left[\frac{\theta^2}{2} \sin(2\theta) + \frac{\theta}{2} \cos(2\theta) - \frac{1}{4} \sin(2\theta) \right]_0^{\pi/4} \\ &= \frac{1}{2} \left[\left(\frac{\pi^2}{32} \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{8} \cos\left(\frac{\pi}{2}\right) - \frac{1}{4} \sin\left(\frac{\pi}{2}\right) \right) - (0 + 0 - 0) \right] \\ &= \frac{1}{2} \left(\frac{\pi^2}{32}(1) + 0 - \frac{1}{4}(1) \right) \\ &= \frac{\pi^2}{64} - \frac{1}{8} \end{aligned}$$

$\frac{\pi^2 - 8}{64}$
