A LEVEL Cambridge Topical Past Papers

FURTHER MATHEMATICS

P1,P2

2020 — 2024

1 - (9231/11_Summer_2020_Q2) - *Roots Of Polynomial Equations*

The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

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(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

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(ii) Find the value of $\alpha^3+\beta^3+\gamma^3$. [2]

2 - (9231/13_Summer_2020_Q1) - *Roots Of Polynomial Equations*

The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

3 - (9231/11_Winter_2020_Q3) - *Roots Of Polynomial Equations, Matrices*

The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^3, β^3, γ^3. [3]

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(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

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(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

4 - (9231/12_Winter_2020_Q1) - *Roots Of Polynomial Equations*

The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

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(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

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(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

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5 - (9231/11_Summer_2021_Q3) - *Roots Of Polynomial Equations*

The equation $x^4 - 2x^3 - 1 = 0$ has roots α, β, γ, δ.

(a) Find a quartic equation whose roots are α^3, β^3, γ^3, δ^3 and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

6 - (9231/13_Summer_2021_Q2) - *Roots Of Polynomial Equations*

The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

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(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

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(ii) Hence, or otherwise, find the value of S_4. [2]

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(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

(d) Find the value of $\frac{1}{\alpha+\beta}+\frac{1}{\beta+\gamma}+\frac{1}{\gamma+\alpha}$. [2]

7 - (9231/11_Winter_2021_Q1) - *Roots Of Polynomial Equations*

It is given that

$$\alpha+\beta+\gamma = 3, \quad \alpha^2+\beta^2+\gamma^2 = 5, \quad \alpha^3+\beta^3+\gamma^3 = 6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

8 - (9231/12_Winter_2021_Q4) - *Roots Of Polynomial Equations*

The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

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(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

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(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

9 - (9231/11_Summer_2022_Q4) - *Roots Of Polynomial Equations*

The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

10 - (9231/11_Winter_2022_Q1) - *Roots Of Polynomial Equations*

The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

11 - (9231/12_Winter_2022_Q2) - *Roots Of Polynomial Equations*

The equation $x^4+3x^2+2x+6=0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$.

[4]

(b) Find the value of $\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}$. [2]

12 - (9231/11_Summer_2023_Q2) - *Roots Of Polynomial Equations*

The cubic equation $x^3 + 4x^2 + 6x + 1 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

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(b) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^2 + (\beta+r)^2 + (\gamma+r)^2\right) = n(n^2 + an + b),$$

where a and b are constants to be determined. [6]

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13 - (9231/13_Summer_2023_Q3) - *Roots Of Polynomial Equations*

The equation $x^4 - x^2 + 2x + 5 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^2, \beta^2, \gamma^2, \delta^2$ and state the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [4]

(b) Find the value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}+\frac{1}{\delta^2}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

14 - (9231/13_Summer_2023_Q7) - *Roots Of Polynomial Equations*

The curve C has equation $y = \frac{x^2 + 2x + 1}{x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the turning points on C. [3]

(c) Sketch C. [3]

(d) Sketch the curves with equations $y = \left|\frac{x^2 + 2x + 1}{x - 3}\right|$ and $y^2 = \frac{x^2 + 2x + 1}{x - 3}$ on a single diagram, clearly identifying each curve. [4]

15 - (9231/11_Winter_2023_Q3) - *Roots Of Polynomial Equations*

The quartic equation $x^4 + bx^3 + cx^2 + dx - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. It is given that

$$\alpha + \beta + \gamma + \delta = 3, \qquad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5, \qquad \alpha^{-1} + \beta^{-1} + \gamma^{-1} + \delta^{-1} = 6.$$

(a) Find the values of b, c and d. [6]

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(b) Given also that $\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = -27$, find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

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16 - (9231/12_Winter_2023_Q4) - *Roots Of Polynomial Equations*

The cubic equation $27x^3 + 18x^2 + 6x - 1 = 0$ has roots α, β, γ.

(a) Show that a cubic equation with roots $3\alpha + 1$, $3\beta + 1$, $3\gamma + 1$ is

$$y^3 - y^2 + y - 2 = 0.$$ [3]

The sum $(3\alpha+1)^n+(3\beta+1)^n+(3\gamma+1)^n$ is denoted by S_n.

(b) Find the values of S_2 and S_3. [4]

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(c) Find the values of S_{-1} and S_{-2}. [3]

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17 - (9231/11_Summer_2024_Q1) - *Roots Of Polynomial Equations*

The cubic equation $2x^3 + x^2 - px - 5 = 0$, where p is a positive constant, has roots α, β, γ.

(a) State, in terms of p, the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [1]

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(b) Find the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. [2]

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(c) Deduce a cubic equation whose roots are $\alpha\beta, \beta\gamma, \alpha\gamma$. [1]

(d) Given that $\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{3}$, find the value of p. [2]

18 - (9231/13_Summer_2024_Q2) - *Roots Of Polynomial Equations*

The cubic equation $x^3 + 2x^2 + 3x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^2 + 1, \beta^2 + 1, \gamma^2 + 1$. [3]

(b) Find the value of $\left(\alpha^2+1\right)^2+\left(\beta^2+1\right)^2+\left(\gamma^2+1\right)^2$. [2]

(c) Find the value of $\left(\alpha^2+1\right)^3+\left(\beta^2+1\right)^3+\left(\gamma^2+1\right)^3$. [2]

19 - (9231/11_Winter_2024_Q3) - *Roots Of Polynomial Equations*

The quartic equation $x^4 + 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^4, \beta^4, \gamma^4, \delta^4$ and state the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [5]

(b) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [3]

(c) Find the value of $\alpha^8 + \beta^8 + \gamma^8 + \delta^8$. [2]

20 - (9231/12_Winter_2024_Q3) - *Roots Of Polynomial Equations*

It is given that

$$\alpha + \beta + \gamma + \delta = 2,$$
$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 3,$$
$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 4.$$

(a) Find the value of $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$. [2]

(b) Find the value of $\alpha^2\beta + \alpha^2\gamma + \alpha^2\delta + \beta^2\alpha + \beta^2\gamma + \beta^2\delta + \gamma^2\alpha + \gamma^2\beta + \gamma^2\delta + \delta^2\alpha + \delta^2\beta + \delta^2\gamma$. [3]

(c) It is given that $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$6x^4 - 12x^3 + 3x^2 + 2x + 6 = 0.$$

(i) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [3]

(ii) Find the value of $\alpha^5 + \beta^5 + \gamma^5 + \delta^5$. [2]

21 - (9231/21_Winter_2024_Q4) - *Matrices, Roots Of Polynomial Equations*

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}.$$

(a) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and state the corresponding eigenvalue. [2]

(b) Show that the characteristic equation of **A** is $\lambda^3 - 19\lambda - 30 = 0$ and hence find the other eigenvalues of **A**. [3]

(c) Use the characteristic equation of **A** to find $\mathbf{A}^{-1}$. [4]

22 - (9231/22_Winter_2024_Q4) - *Complex Numbers, Roots Of Polynomial Equations*

(a) Use de Moivre's theorem to show that

$$\cot 6\theta = \frac{\cot^6\theta - 15\cot^4\theta + 15\cot^2\theta - 1}{6\cot^5\theta - 20\cot^3\theta + 6\cot\theta}.$$ [6]

(b) Hence obtain the roots of the equation

$$x^6 - 6x^5 - 15x^4 + 20x^3 + 15x^2 - 6x - 1 = 0$$

in the form $\cot(q\pi)$, where q is a rational number. [4]

23 - (9231/23_Winter_2024_Q1) - *Roots Of Polynomial Equations*

Find the set of values of k for which the system of equations

$$\begin{aligned} x + 5y + 6z &= 1, \\ kx + 2y + 2z &= 2, \\ -3x + 4y + 8z &= 3, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [4]

1 - (9231/11_Summer_2020_Q1) - *Rational Functions And Graphs*

Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$. [4]

2 - (9231/11_Summer_2020_Q3) - *Rational Functions And Graphs*

The curve C has equation $y = \dfrac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

(c) Sketch C. [3]

3 - (9231/13_Summer_2020_Q6) - *Rational Functions And Graphs*

The curve C has equation $y = \frac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\frac{10+x-2x^2}{2x-3}\right| < 4$. [6]

4 - (9231/11_Winter_2020_Q6) - *Rational Functions And Graphs*

The curve C has equation $y = \frac{x^2 + x - 1}{x - 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 + x - 1}{x - 1}\right|$. [2]

5 - (9231/12_Winter_2020_Q6) - *Rational Functions And Graphs*

Let a be a positive constant.

(a) The curve C_1 has equation $y = \frac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\frac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\frac{x-a}{x-2a}\right|$.

(b) **(i)** Find the coordinates of any stationary points of C_2. [3]

(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

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(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]

6 - (9231/11_Summer_2021_Q7) - *Rational Functions And Graphs*

The curve C has equation $y = \dfrac{x^2 + x + 9}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the coordinates of the stationary points on C. [4]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 + x + 9}{x + 1}\right|$ and find the set of values of x for which $2|x^2 + x + 9| > 13|x + 1|$. [5]

7 - (9231/13_Summer_2021_Q7) - *Rational Functions And Graphs*

The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]

8 - (9231/11_Winter_2021_Q7) - *Rational Functions And Graphs*

The curve C has equation $y = \dfrac{4x+5}{4-4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\frac{4x+5}{4-4x^2}\right|$ and find in exact form the set of values of x for which $4|4x+5| > 5|4-4x^2|$. [6]

9 - (9231/12_Winter_2021_Q6) - *Rational Functions And Graphs*

The curve C has equation $y = \frac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that there is no point on C for which $0 < y < 12$. [4]

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(c) Sketch C. [2]

(d) **(i)** Sketch the graphs of $y = \left|\frac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(ii) Use your sketch to find the set of values of c for which $\left|\frac{x^2}{x-3}\right| \leqslant |x| + c$ has no solution. [1]

10 - (9231/11_Summer_2022_Q5) - *Rational Functions And Graphs*

The curve C has equation $y = \frac{2x^2 - x - 1}{x^2 + x + 1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

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(b) Find the coordinates of the stationary points on C. [4]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{2x^2 - x - 1}{x^2 + x + 1}\right|$ and state the set of values of k for which $\left|\frac{2x^2 - x - 1}{x^2 + x + 1}\right| = k$ has 4 distinct real solutions. [2]

11 - (9231/13_Summer_2022_Q1) - *Rational Functions And Graphs*

(a) Sketch the curve with equation $y = \frac{x+1}{x-1}$. [2]

(b) Sketch the curve with equation $y = \frac{|x|+1}{|x|-1}$ and find the set of values of x for which $\frac{|x|+1}{|x|-1} < -2$. [4]

12 - (9231/13_Summer_2022_Q3) - *Rational Functions And Graphs*

A curve C has equation $y = \dfrac{ax^2 + x - 1}{x - 1}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that there is no point on C for which $1 < y < 1 + 4a$. [4]

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(c) Sketch C. You do not need to find the coordinates of the intersections with the axes. [3]

13 - (9231/11_Winter_2022_Q7) - *Rational Functions And Graphs*

The curve C has equation $y = \frac{5x^2}{5x-2}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the coordinates of the stationary points on C. [4]

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(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \left|\frac{5x^2}{5x-2}\right|$ and find in exact form the set of values of x for which $\left|\frac{5x^2}{5x-2}\right| < 2$. [6]

14 - (9231/12_Winter_2022_Q7) - *Rational Functions And Graphs*

The curve C has equation $y = \dfrac{x^2 - x}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the exact coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 - x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x}{x+1}\right| < 6$. [5]

ANSWERS

1 - (9231/11_Summer_2020_Q2) - *Roots Of Polynomial Equations*

(a)	$y = x^2$	B1
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py+5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	M1
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	A1
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(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \dfrac{p^2+36}{36}$	B1
	$\dfrac{p^2+36}{36} = -\dfrac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	M1
	$p = -6$	A1
		3
(b)(ii)	$6(\alpha^3+\beta^3+\gamma^3) = 6(\alpha^2+\beta^2+\gamma^2) + 3(\alpha+\beta+\gamma) + 15$	M1
	$\alpha^3+\beta^3+\gamma^3 = 5$	A1
		2

2 - (9231/13_Summer_2020_Q1) - *Roots Of Polynomial Equations*

(a)	$y = x^{-1}$	B1
	$7y^{-3} + 3y^{-2} + 5y^{-1} + 1 = 0 \Rightarrow y^3 + 5y^2 + 3y + 7 = 0$	M1 A1
		3
(b)	$\alpha^{-2} + \beta^{-2} + \gamma^{-2} = (-5)^2 - 2(3) = 19$	M1 A1
(c)	$\alpha^{-3} + \beta^{-3} + \gamma^{-3} = -5(19) - 3(-5) - 21 = -101$	M1 A1
		4

3 - (9231/11_Winter_2020_Q3) - *Roots Of Polynomial Equations, Matrices*

(a)	$y = x^3 \Rightarrow x = y^{\frac{1}{3}}$	**B1**	Substitutes.
	$y + cy^{\frac{1}{3}} + 1 = 0 \Rightarrow -c^3 y = (y+1)^3 = y^3 + 3y^2 + 3y + 1$	**M1**	Correct attempt to eliminate cube root.
	$y^3 + 3y^2 + (3 + c^3)y + 1 = 0$	**A1**	
		3	
(b)	$\alpha^3 + \beta^3 + \gamma^3 = -3 \quad \alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 = 3 + c^3$	**B1 FT**	Using *their* answer to **(a)**.
	$\alpha^6 + \beta^6 + \gamma^6 = (-3)^2 - 2(3 + c^3)$	**M1**	$\alpha^6 + \beta^6 + \gamma^6 = (\alpha^3 + \beta^3 + \gamma^3)^2 - 2(\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3)$
	$= 3 - 2c^3$	**A1**	AG
		3	
(c)	$\alpha^3\beta^3\gamma^3 = -1$	**B1**	If using *their* answer to **(a)** FT
	$\begin{vmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{vmatrix} = 1 - (\alpha^6 + \beta^6 + \gamma^6) + 2\alpha^3\beta^3\gamma^3 = 2c^3 - 4$	**M1 A1**	Evaluates determinant.
	$2c^3 - 4 = 0$	**M1**	Sets determinant equal to zero.
	$c = \sqrt[3]{2}$	**A1**	
		5	

4 - (9231/12_Winter_2020_Q1) - *Roots Of Polynomial Equations*

(a)	$d = 1$	**B1**	
		1	
(b)	$y = x + 1 \Rightarrow x = y - 1$	**B1**	Uses correct substitution.
	$y^3 + (b-3)y^2 + (c - 2b + 3)y + b - c = 0$	**M1 A1**	Substitutes and expands.
	Alternative method for question 1(b)		
	$(\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\beta+1)(\gamma+1) = c - 2b + 3$	**B1**	
	$(\alpha+1+\beta+1+\gamma+1) = 3 - b, \ (\alpha+1)(\beta+1)(\gamma+1) = c - b$	**M1**	Using these relationships.
	$y^3 + (b-3)y^2 + (c - 2b + 3)y + b - c = 0)$	**A1**	
		3	
(c)	$\beta + 1 = -(b - 3)$	**B1**	Uses sum of roots.
	$-(\alpha+1)(\alpha+1) = c - 2b + 3$	**B1**	Uses sum of products in pairs.
	$-(\alpha+1)(\beta+1)(\alpha+1) = -(b - c)$	**M1**	Applies product of roots.
	$\Rightarrow (c - 2b + 3)(b - 3) = b - c$	**A1**	AG
		4	

5 - (9231/11_Summer_2021_Q3) - *Roots Of Polynomial Equations*

(a)	$y = x^3$	**B1**	Correct substitution.
	$y^{\frac{4}{3}} - 2y - 1 = 0 \Rightarrow y^4 = (2y+1)^3 = 8y^3 + 12y^2 + 6y + 1$	**M1**	Obtains an equation not involving radicals.
	$y^4 - 8y^3 - 12y^2 - 6y - 1 = 0$	**A1**	
	$\alpha^3 + \beta^3 + \gamma^3 + \delta^3 = 8$	**B1 FT**	
		4	

(b)	$\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}=\frac{\alpha^3\beta^3\delta^3+\alpha^3\beta^3\gamma^3+\beta^3\gamma^3\delta^3+\alpha^3\gamma^3\delta^3}{\alpha^3\beta^3\gamma^3\delta^3}=\frac{6}{-1}$	**M1 A1 FT**	Relates $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$ to coefficients.
	-6	**A1**	
		3	
(c)	$\alpha^4+\beta^4+\gamma^4+\delta^4=2(\alpha^3+\beta^3+\gamma^3+\delta^3)+4$	**M1**	Uses original equation.
	$=20$	**A1**	
		2	

6 - (9231/13_Summer_2021_Q2) - *Roots Of Polynomial Equations*

(a)	$S_1=2$	**B1**	
	$S_2=S_1^2-2(0)$	**M1**	Uses formula for sum of squares.
	$=4$	**A1**	Correct answer implies M1A1.
		3	
(b)(i)	$S_{n+3}=2S_{n+2}-\frac{3}{2}S_n$	**B1**	CAO or as a single fraction.
		1	
(b)(ii)	$S_4=2S_3-\frac{3}{2}S_1=2(2S_2-\frac{3}{2}S_0)-\frac{3}{2}S_1$	**M1**	Uses their recursive formula from part **(i)** to find S_4 $\left[S_3=\frac{7}{2}\right]$.
	$=4$	**A1**	
		2	
(c)	$x=2-y$	**B1**	SOI
	$2(2-y)^3-4(2-y)^2+3=0$	**M1**	Makes *their* substitution.
	$2y^3-8y^2+8y-3=0$	**A1**	OE but must be an equation.
		3	
(d)	$\frac{\frac{8}{2}}{\frac{-3}{2}}$ OR Or use $2S_2-8S_1+8S_0-3S_{-1}=0$ with substitution of their values	**M1**	Uses $\frac{1}{\alpha'}+\frac{1}{\beta'}+\frac{1}{\gamma'}=\frac{\alpha'\beta'+\beta'\gamma'+\gamma'\alpha'}{\alpha'\beta'\gamma'}$.
	$=\frac{8}{3}$	**A1 FT**	FT from **2(c).**
		2	

7 - (9231/11_Winter_2021_Q1) - *Roots Of Polynomial Equations*

$b=-(\alpha+\beta+\gamma)=-3$	**B1**	
$5=3^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$	**M1 A1**	Uses formula for sum of squares.
$c=2$	**A1**	
$6-3(5)+2(3)+3d=0$	**M1**	Uses original equation or formula for sum of cubes.
[Equation is $x^3-3x^2+2x+1=0$] $d=1$	**A1**	
	6	